Population Dynamics: Theory of Nonstable Populations

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Abstract

This article introduces the demographic theory of nonstable populations. The analysis of population dynamics was originally a branch of mathematical biology but is also important for social demography because it provides elegant models for how populations ebb and flow.

Background

To introduce the theory of nonstable population dynamics, we start by presenting a Malthusian model studied by Lee (1974). Consider an overlapping-generation framework in which each individual lives one or two periods. The first period is childhood and the second is adulthood, and all surviving adults will be in the labor force. The Malthusian model can be characterized by the following two equations: $W_t = f(N_t)$, $b_t = g(W_t)$, where W_t is the wage rate (at time t), N is the size of the adult group, b is the crude birth rate, the $f(.)$ function characterizes the wage/employment relationship in the labor market, and the $g(.)$ function specifies the influence of the labor market reward on fertility.

Suppose the number of children born in period t is B_t, and the child survival rate is l, then $N_t = l \cdot B_{t-1}$. The above two equations can be combined in the following reduced form: $B_t = h(B_{t-1})$, where $h(B_t) = g(f(l \cdot B_{t-1})) \cdot l \cdot B_{t-1}$. The above expression is a recursive equation of B_t, from which the dynamic pattern can be analyzed, and the possible existence of cyclicity can be studied. Here, the Malthusian model is interpreted as a model of density dependency, where the density is characterized by the birth size of the previous period. It turns out that the Malthusian model described above may not converge to a 'dismal' steady state, as Malthus suggested. Mathematically, the B_t series may converge, diverge, or be cyclic, and the volatility of the birth series B_t crucially hinges on the feedback elasticity of $h(.)$ with respect to B_{t-1}. When this elasticity is sufficiently large, the B_t series can easily have cycles. This is the classical case of nonstable population. Modern versions of nonstable population will be discussed below, after we introduce the general population structure.

The General Population Structure

Let $N_t \equiv (N_{t,\ 1}, \ldots, N_{t,\ n})$ be an n-type population vector at period t, where $N_{t,\ i}$, $i = 1, \ldots, n$ indicates the size of type-i population at period t. The type here may refer to any criterion that is used to classify the population, such as age, sex, and occupation. A general formulation of population dynamics can be written as follows:

$$N_{t+1} = Q(N_t)N_t \qquad [1]$$

where $Q(N_t)$ is the transition matrix. The simplest case is when $Q(N_t) = Q$, a time-independent matrix. In this situation, the above equation reduces to a time-invariant Markov chain: $N_{t+1} = QN_t$. It is a well-known implication of the Frobenius–Perron theorem that if Q is positively regular, meaning that there is flexible cross-period mobility across types, then the population vector N_t will converge to a steady state, in which all elements of the N_t vector grow at the same exponential rate, thereby the population composition is also time invariant. It has been shown by Chu (1998) that, for most population models with time-independent Q, this positive regularity condition is likely to be satisfied, therefore any nonstable behavior of the population would appear only when $Q(N_t)$ is not a constant matrix, or equivalently when the relationship between N_{t+1} and N_t in eqn [1] has some nonlinearity. Below we will introduce several special cases of the generic setting in eqn [1], and discuss their empirical relevance. The reader can see that each model corresponds to a special interpretation of the generic setting in eqn [1].

Age-Structured Cycles

For demographers, the most familiar version of the dynamic models in eqn [1] is the Lotka–Leslie age-specific models, where the type criterion is age. Let $N_{t,\ i}$ be the number of population aged i at period t, B_{t-i} the birth size at period $t-i$, and l_i probability that a person can survive to age i, then the following relationship must hold: $N_{t,\ i} = B_{t-i} \cdot l_i$. The transformed dynamics in terms of B_t is the well-known Lotka's equation:

$$B(t) = \sum_i B_{t-i} \cdot l_i \cdot m_{t,i} = \sum_i B_{t-i} \cdot \phi_{t,i} \qquad [2]$$

where $m_{t,\ i}$ is the average number of births per surviving member aged i at period t, and $\phi_{t,\ i} \equiv l_i \cdot m_{t,\ i}$ is the net maternity function. Evidently, the Malthusian model presented in the previous section is a special case of eqn [2] with two age types. As we mentioned, when the fertility function $m_{t,\ i}$ is independent of t, then the process of B_t follows a time-invariant Markov chain, and the usual convergence result applies. But if $m(t, i)$ is time dependent, then cycles or more volatile behavior may appear. One typical case of such time dependency is when the age-specific fertility rates are functions of previous birth sizes $m_{t,\ i} = m_i \cdot (B_t)$, where $B_t \equiv (B_{t-1}, B_{t-2}, \ldots)$.

Suppose the birth series corresponding to eqn [2] has a steady state. Taking a Taylor expansion around this steady state, Lee (1974) showed that the volatility of the birth series characterized by eqn [2] hinges upon the elasticities of the net maternity function with respect to B_t. When previous birth sizes are larger, usually the density pressure is larger. Intuitively, larger feedback elasticities imply that present reproduction would be more sensitive to previous birth shocks; hence the birth series would be more volatile. Easterlin (1961) argued that the baby-boom/baby-bust cycles observed in the United States are typical examples of a feedback cycle.

How large the feedback elasticities are is an empirical question. In order to estimate such elasticities, two models have been proposed: The first is to assume that $\phi_{t,\ i}$ is only a function of the cohort size aged i: $\phi_{t,\ i}=\phi_i\cdot(B_{t-i})$ and the second is to assume that $\phi_{t,\ i}$ is a function of the weighted average of birth sizes of different ages $\phi_{t,\ i}=\phi_i\cdot(\sum_j w_j\cdot B_{t-j})$. The former is referred to as the cohort model, and the latter as the period model. However, empirical evidence has so far failed to back up either of these two models in producing persistent birth cycles that fit all characteristics (such as amplitude and period). Moreover, even if there does exist a cyclic solution to eqn [2], the information concerning feedback elasticity is not sufficient for us to tell whether the cyclical solution in question is stable. Related analysis is rather technical, and the reader is referred to Tuljapurkar (1987) for details.

Predator–Prey Cycles

The predator–prey model describes a two-type population structure, where the types are prey and predator, respectively, referred to as type-1 and type-2. The simplest formulation of the predator–prey model is characterized by the following difference equations:

$$\begin{aligned}\Delta N_{t+1,1} &\equiv N_{t+1,1}-N_{t,1}\\ &= aN_{t,1}-bN_{t,1}N_{t,2}\\ \Delta N_{t+1,2} &\equiv N_{t+1,2}-N_{t,2}\\ &= -cN_{t,2}+dN_{t,1}N_{t,2},\quad a,b,c,d>0\end{aligned} \qquad [3]$$

Equation [3] is obviously a special case of eqn [1]. The prey is considered the only food resource available to the predator. Thus, if $N_{t,\ 1}=0$, the predator population decreases exponentially at the rate C. With the nonexistence of predators, the prey population grows exponentially at the rate a.

In the steady state of eqn [3], $\Delta N_{t,\ 1}=\Delta N_{t,\ 2}=0$, there is only one nontrivial solution: $(N_1^*,N_2^*)=(c/d,a/b)$. Let the Jacobian matrix of eqn [3] around the nontrivial solution be denoted J. It turns out that the J matrix can be written as

$$J=\begin{pmatrix} a-bN_2^* & -bN_1^* \\ dN_2^* & -c+dN_1^* \end{pmatrix}=\begin{pmatrix} 0 & -bc/d \\ ad/b & 0 \end{pmatrix}$$

Hence, we have $|J|=ac>0$ and the trace of J is zero.

Thus, we know that the eigenvalues of J are purely imaginary, meaning that besides the equilibrium point $(c/d,a/b)$ itself, the solution trajectories of eqn [3] are cyclic and nonstable. But it was also pointed out that when the predator–prey model embodies the concerns of diminishing or increasing returns, then the solution becomes locally stable or divergent.

Occupation–Switch Cycles

Based on stylized facts found in Chinese history, Chu and Lee (1994) proposed an occupation-specific population model, in which the population is separated into three types: peasants, soldiers, and bandits. Peasants grow crops and pay taxes, soldiers (rulers) collect taxes and hunt for bandits, and bandits rob food and clash with peasants and soldiers. It is assumed that soldiers are drafted by the government, and all other civilians can choose to become peasants or bandits. Their occupation choice is made depending on which occupation is expected to generate a higher utility. When the bandit/soldier ratio is relatively high, it is likely that the incumbent power regime will be overthrown by the bandit group, and the dynasty switches. It turns out that this pattern of population dynamics characterizes the dynamic evolution of dynasties in Chinese history, and the cycles were called dynastic cycles by historians.

There is a unique feature associated with the above-mentioned occupation-switching cycles. For most animals, it is believed that when density pressure occurs, the environment becomes less favorable, and the reproduction rate of animals is reduced. One might expect that rational human decisions may be able to weaken the outside density pressure through institutional and rational regulations. But the scenario of dynastic cycles seems to be a counterexample.

Human beings' occupational choice between peasants and bandits has a 'demonstration effect,' which tends to magnify the originally weak density pressure and 'destablize' the dynamics of the population compositional structure.

Laborer–Capitalist Cycles

The cyclical pattern of population composition mentioned above is not the only case we find in human societies. Some researchers have considered a neoclassical economic growth model, and focused on the two economic classes in the economy: the laborers and the capitalists. Let u_t be the income share attributed to labor, and v_t be the employment rate of labor. Under some reasonable assumptions, Goodwin (1967) has shown that the dynamics of u_t and v_t mimic that of eqn [3]. Thus, the conclusion that the capitalists' economy appears to be 'permanently oscillating' was reached. But just as the original predator–prey model is sensitive to variations in institutional specifications, Goodwin's model after some minor modification also generates qualitatively different results. Definite evidence of the nonstability of human population is yet to be found.

Human Development Cycles

The final population dynamics we discuss is related to the economic development process of human beings. The Neolithic Revolution about 10 000 years ago triggered the human society from hunter-gatherer to agricultural. The Industrial Revolution 250 years ago marked the beginning of the modern phase. In terms of population dynamics, we observe that in the agricultural epoch the population was dominated by the Malthusian rule with a stable population size and a constant per-capita income, whereas after we entered the industrial revolution we first experienced a demographic transition and then a near-replacement level population growth.

The demographic transition changes parents' choices from quantity to quality of children, and the increased level of child education leads to the growth of per capita income. In short, we have a subsistence level family income and stagnant population size 300 years ago, and a growing population size (estimated 9.3 billion by 2050) with drastically different income levels across countries. What is behind this big regime change of dynamics?

It was the contribution of Galor (2011) to have explained the above-mentioned regime change in a unified model. Among others, there are two factors that dominated the change of population dynamics. The first is the binding subsistence level of consumption in the Malthusian period, when parents cannot spare their resources to increase children's quality. The second is the improved life expectancy and the change of market/household comparative advantage, which facilitate women to adopt a quantity–quality trade-off. Countries in the modern world are therefore divided into pre-transition, transition-leaders and transition-followers. The first category has a fast population growth but yet a low income level, and the latter two categories have a stable population size and a growing income level.

See also: Demographic Models; Population Dynamics: Classical Applications of Stable Population Theory; Population Dynamics: Theory of Stable Populations.

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