

Population projection

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Population projection

- Transition matrices
- Structural zeros
- The Leslie matrix subdiagonal
- The Leslie matrix first row
- Projecting fillies, mares, seniors



Transition matrices

- Transition matrices are tables used for population projection
- Official presentations of projections are often filled with disclaimers cautioning the reader that **projections are not predictions**
 - They do not tell us what the world **will** be like but only what the world **would** be like if a particular set of stated assumptions about future vital rates turned out to be true
- The assumptions may or may not bear any relation to what actually happens



Disingenuous disclaimers

- Projection is not just a game with computers and pieces of paper
- We do projections for a purpose to foresee the future of the population
- The choice of credible assumptions about vital rates is an important part of the art and science of projection
 - As are the formulas we use to implement the calculations
 - In this chapter, we concentrate on the formulas



Previous failures

- The record of demographers at guessing future vital rates has not been good
- The community failed to predict
 - The Baby Boom of the 1950s and 1960s
 - The Baby Lull of the 1970s and 1980s
 - It largely failed to predict the continuing trend toward lower mortality at older ages in industrialized countries
- We do not yet understand the mechanisms that drive demographic change
 - We need deeper theories with better predictive power

But we are doing good

- Despite these failures with previous predictions, demographers do better than economists, seismologists...
- Choice of assumptions about future fertility, mortality, marriage, divorce, and immigration may be difficult
- But methods for using those assumptions to calculate future population sizes and age distributions are well developed and satisfactory

Focus of this chapter

- We study these methods of calculation
- Tools based on matrices and vectors
- Projection over discrete steps of time
- Populations split up into discrete age groups



Several characteristics

- Sophisticated projections can treat a population classified by many characteristics
 - Sex, age, race, ethnicity, education, marital status, income, locality
 - So much detail is not common
- Progress is being made, under a European team led by the demographer Wolfgang Lutz
 - Incorporating education into worldwide projections

Basic ideas: simple case

- The basic ideas are illustrated by simple projections
- Focus on a single sex
- All races and ethnicities together
- We subdivide the population only by age



Leslie matrices

- The main tools for projecting the size and age distribution of a population forward through time are tables called “Leslie matrices”
 - By P.H. Leslie (1945)
 - Same approach done few years earlier by H. Bernardelli and E.G. Lewis
- Related to Markov chains in probability theory
 - But what is being projected with Leslie matrices are expected numbers of individuals rather than probabilities



Leslie matrices for age structure

- Leslie matrices are a special case of transition matrices
- Demographers use general transition matrices
 - To project a distribution of marital status, parity, education, or other variables into the future
- They use the special transition matrices that Leslie defined
 - To project age structure



Defining a transition matrix

- A transition matrix is a table with rows and columns showing the expected number of individuals
 - who **end** up in the state with the label on the **row**
 - per individual at the **start** in the state with the label on the **column**

	0 to 5	5 to 10	10 to 15	15 to 20	20 to 25
0 to 5	kids	kids	kids	kids	kids
5 to 10	survivors	0	0	0	0
10 to 15	0	survivors	0	0	0
15 to 20	0	0	survivors	0	0
20 to 25	0	0	0	survivors	0



One-step transition

- A Leslie matrix is a special case of a transition matrix in which the states correspond to age groups
 - Processes of transition are surviving and giving birth
- The Leslie matrix describes a one-step transition
 - We project the population forward one step at a time
 - The time between **start** and **end** (projection step) should be equal to the **width** (n) of all age groups



Age group width

- The fact that the step size has to equal the age group width is crucial
 - Generally pick one sex, usually females
 - Divide the female population into age groups of width n
 - Width may be 1, 5, 15 years, or some other number

Examples

- If we are using 1-year age groups
 - We have to project forward 1 year at a time
 - To project 10 years into the future requires 10 projection steps
- If we are using 5-year age groups
 - We have to project forward 5 years at a time
 - To project 10 years into the future requires only 2 projection steps



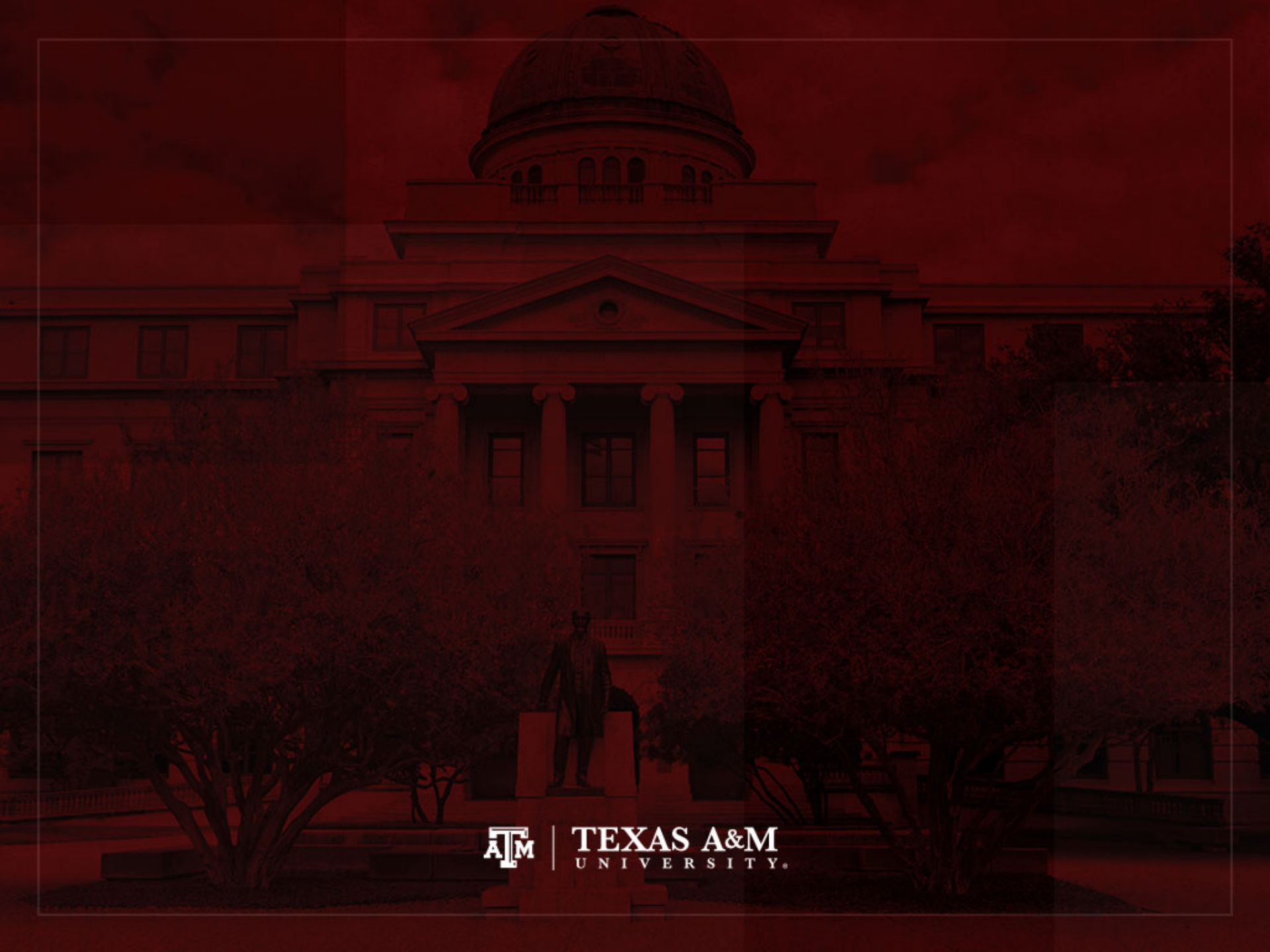
Closed population

- We assume a closed population
- No migrants are included in projections
- People enter the population only by being born to members already in the population
- People leave it only by dying



Childbirth as a transition

- Projections treat childbirth as a possible transition along with survival
 - People who end up in some state may not be the same people who start in any one of the states
 - They may be the babies of people who start in the various states
- We are concerned with the expected numbers
 - In the state for the row (end)
 - Per person in the state for the column (start)
 - Without regard to how the people are channeled there



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Structural zeros

- The logic of the transition process is built into a transition matrix through the pattern of zeros
- Some age groups owe no part of their numbers at the end of the step to certain other age groups
- Suppose we have $n=5$
 - We have 5-year-wide age groups
 - We are projecting forward 5 years in one step



Example of teenagers

- No teenagers owe their numbers to 40-year-olds five years before
- The value of the Leslie matrix element has to be zero in
 - The row for 15 to 20-year-olds (end)
 - The column for 40-year-olds (start)
- This is a “structural zero”
 - We know because of the logic of the processes of aging and childbirth



Example of a Leslie matrix

- Most of the elements of a Leslie matrix are structural zeros
 - We can fill them in immediately
 - At the end of 5 years, people age up to 5 years
 - No one can get younger over time

Start

		0 to 5	5 to 10	10 to 15	15 to 20	20 to 25
End	0 to 5	kids	kids	kids	kids	kids
	5 to 10	survivors	0	0	0	0
	10 to 15	0	survivors	0	0	0
	15 to 20	0	0	survivors	0	0
	20 to 25	0	0	0	survivors	0



Following the same logic

- Below the first row, all elements are structural zeros except the subdiagonal
- No one can
 - Jump an age group
 - Stay in the same age group
 - Get younger
- They can only move into the next age group
 - If they survive
 - It is important to have the same age-group width



First row

- What about the first row, for people who end up aged 0 to 5 at the end of 5 years?
 - No one can survive into this row
 - These elements are not structural zeros
 - There can be babies born during the projection step who are found in this age group at the end of the step
 - The number of babies depends on the number of potential parents in the various age groups at the start

First row of a Leslie matrix

Potential parents at the start

**Babies
at the end**

	0 to 5	5 to 10	10 to 15	15 to 20	20 to 25
0 to 5	kids	kids	kids	kids	kids
5 to 10	survivors	0	0	0	0
10 to 15	0	survivors	0	0	0
15 to 20	0	0	survivors	0	0
20 to 25	0	0	0	survivors	0

Upper-left element

- The upper-left element equals zero depending on the age-group width
 - If $n=5$, we do not expect there to be any babies in 5 years to people 0 to 5 at the start
 - If $n=15$, we do expect babies in the next 15 years to people aged 0 to 15 at the start
- This element also depends on empirical knowledge about youngest ages of childbearing
 - It is often equal to zero
 - But it is not regarded as a structural zero



Representing structural zeros

- Another way of representing information about structural zeros is a diagram of permitted transitions
- We mark states within circles and draw an arrow from one state to another if there is a nonzero element for that column-row pair
 - Show links from individuals in the sender state at the beginning of the arrow
 - To individuals who can show up in the receiver state at the end of the arrow

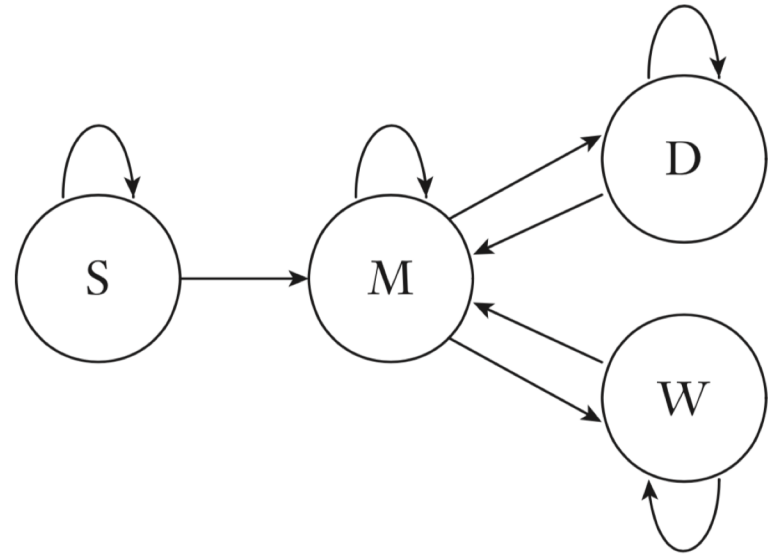


Arrow diagrams

- Arrow diagrams are helpful when transitions are not between age groups
- Useful for transitions between states with a logic of their own
- Same idea as programmer flow charts

Example for marital status

- Four states
 - Single (S), never married
 - Married (M)
 - Widowed (W)
 - Divorced (D)

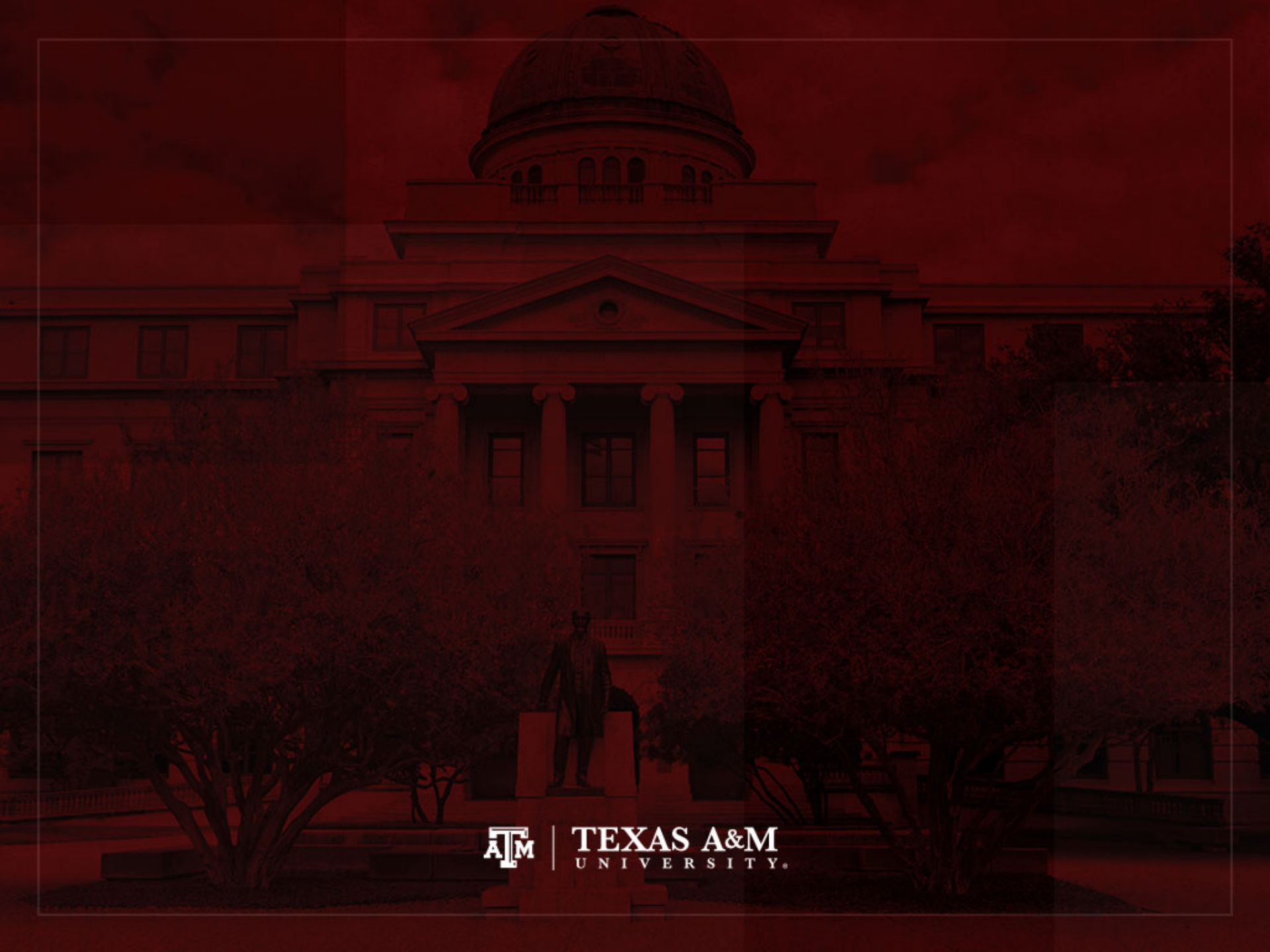


- Suppose the projection step is too short
 - Nobody can get both married and divorced, or both divorced and remarried within a single step
 - Multiple transitions within one step are not numerically significant

Marital status transition matrix

- The structural zeros in the transition matrix corresponding to the previous diagram appear in slots marked “0”

	Single	Married	Widowed	Divorced
Single	x	0	0	0
Married	x	x	x	x
Widowed	0	x	x	0
Divorced	0	x	0	x



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The Leslie matrix subdiagonal

- We generally denote a matrix by a single capital letter like A
 - The first subscript is for the row and the second subscript is for the column
 - $A_{3\ 2}$ element in third row and second column, survivors from the second age group to the third age group
- This notation for matrix elements is universal
 - $A_{\text{to, from}}$ or $A_{\text{row, column}}$
 - Subscript for the destination age group comes first
 - Subscript for the origin age group comes second



Developing a formula

- Develop a formula for the elements along the subdiagonal of the Leslie matrix
 - They represent transitions of survival
 - Continue to use 5-year-wide age groups

Start

		0 to 5	5 to 10	10 to 15	15 to 20	20 to 25
End	0 to 5	kids	kids	kids	kids	kids
	5 to 10	survivors	0	0	0	0
	10 to 15	0	survivors	0	0	0
	15 to 20	0	0	survivors	0	0
	20 to 25	0	0	0	survivors	0



Example

- Consider $A_{3\ 2}$, the expected number of people
 - Aged 10 to 15 at the end (row)
 - Per person aged 5 to 10 at the start (column)
- Age group of 5-to-10-year-olds at the start, at time t (“today”)
 - Composed of cohorts born between times $t-10$ and $t-5$
- We follow the experience of five 1-year birth cohorts on the Lexis diagram...



Age

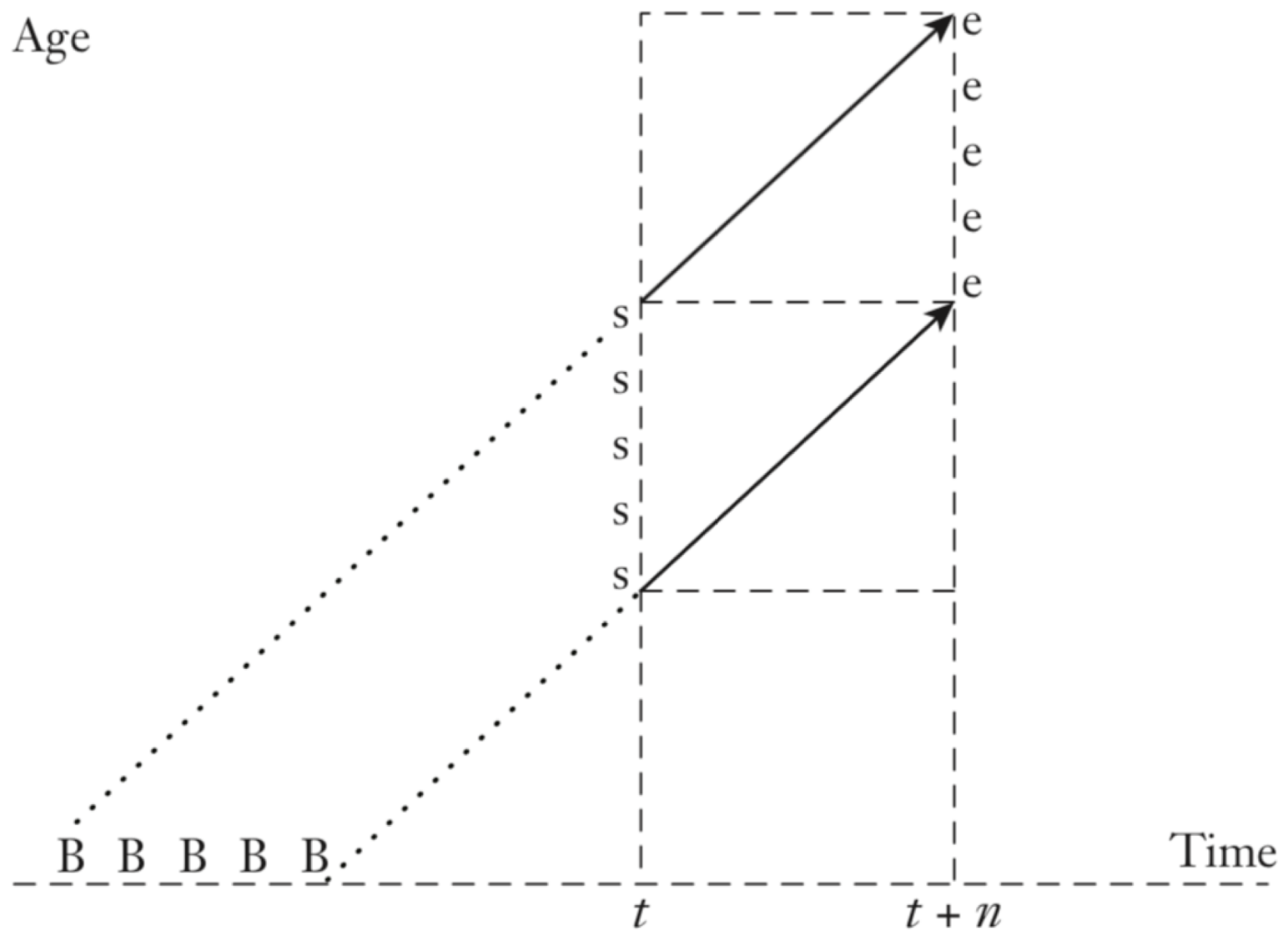


Figure 5.2 Contributions to the Leslie matrix subdiagonal



Symbols

- “B” symbols represent single-year cohorts at birth
- “s” symbols represent cohorts at the start of the projection step
 - 5–10 age group
- “e” symbols represent cohorts at the end of the projection step
 - 10–15 age group

Assumptions

- We ignore changes in sizes of cohorts at birth inside each 5-year period
 - We pretend that all changes in initial cohort sizes occur in jumps between periods
- We also assume that the same lifetable (mortality)
 - Applies to all the 1-year cohorts in this 5-year group of cohorts
 - From 5–10 age group to 10–15 age group



Cohorts in the diagram

- Examples of some of the cohorts on the diagram
- The earliest/oldest cohort (now 10-year-olds)
 - They had some size l_0 at birth
 - At the start of the projection step (s), l_{10} members left
 - At the end of the projection step (e), l_{15} members left
- The latest/youngest cohort (now 5-year-olds)
 - They had the same size l_0 at birth
 - Now (s), l_5 members left
 - Five years from now (e), l_{10} members left

Size of 5-year age group

- The whole 5-year age group is about five times as large as the average size of its youngest and oldest cohorts

- At the start of the projection step (s): t

$$(5/2) (l_5 + l_{10})$$

- At the end of the projection step (e): $t + n$

$$(5/2) (l_{10} + l_{15})$$



Person-years lived

- The sizes of the 5-year age group at the start and end are approximations for person-years lived

$${}_5L_5 = (5/2) (l_5 + l_{10})$$

$${}_5L_{10} = (5/2) (l_{10} + l_{15})$$

- If the age group was split into many small cohorts
 - We could add them all up
 - We would have obtained the ${}_nL_x$ values exactly



Subdiagonal as ratios

- The subdiagonal element $A_{3\ 2}$ of the Leslie matrix
 - Ratio of the 10–15-year-olds at the end
 - To the 5–10-year-olds at the start

$$A_{3\ 2} = \frac{{}_5L_{10}}{{}_5L_5}$$

	0 to 5	5 to 10	10 to 15	15 to 20	20 to 25
0 to 5	kids	kids	kids	kids	kids
5 to 10	$\frac{{}_5L_5}{{}_5L_0}$	0	0	0	0
10 to 15	0	$\frac{{}_5L_{10}}{{}_5L_5}$	0	0	0
15 to 20	0	0	$\frac{{}_5L_{15}}{{}_5L_{10}}$	0	0
20 to 25	0	0	0	$\frac{{}_5L_{20}}{{}_5L_{15}}$	0



Ratio of L values

- Each subdiagonal element of a Leslie matrix is a ratio of big- L values
 - The age label on the **numerator** comes from the **row**
 - The age label on the **denominator** comes from the **column**

$$\begin{array}{l} \phantom{0 \text{ to } 5} \\ 0 \text{ to } 5 \\ 5 \text{ to } 10 \\ 10 \text{ to } 15 \\ 15 \text{ to } 20 \\ 20 \text{ to } 25 \end{array} \begin{pmatrix} 0 \text{ to } 5 & 5 \text{ to } 10 & 10 \text{ to } 15 & 15 \text{ to } 20 & 20 \text{ to } 25 \\ \text{kids} & \text{kids} & \text{kids} & \text{kids} & \text{kids} \\ \frac{{}_5L_5}{{}_5L_0} & 0 & 0 & 0 & 0 \\ 0 & \frac{{}_5L_{10}}{{}_5L_5} & 0 & 0 & 0 \\ 0 & 0 & \frac{{}_5L_{15}}{{}_5L_{10}} & 0 & 0 \\ 0 & 0 & 0 & \frac{{}_5L_{20}}{{}_5L_{15}} & 0 \end{pmatrix}$$



General notation

- The bottom age (denominator) of the age group x is expressed as $x = jn - n$
 - In terms of the column number j

$$A_{j+1, j} = \frac{{}_n L_{x+n}}{{}_n L_x}$$

- $A_{3\ 2}$: from 5–10 (denominator) to 10–15 (numerator)
 - $x = jn - n = 2(5) - 5 = 10 - 5 = 5$

$$A_{3\ 2} = \frac{{}_5 L_{10}}{{}_5 L_5}$$



Different cohorts

- The cohort born between $t-x-n$ and $t-x$ supplies the numerator and denominator in this element

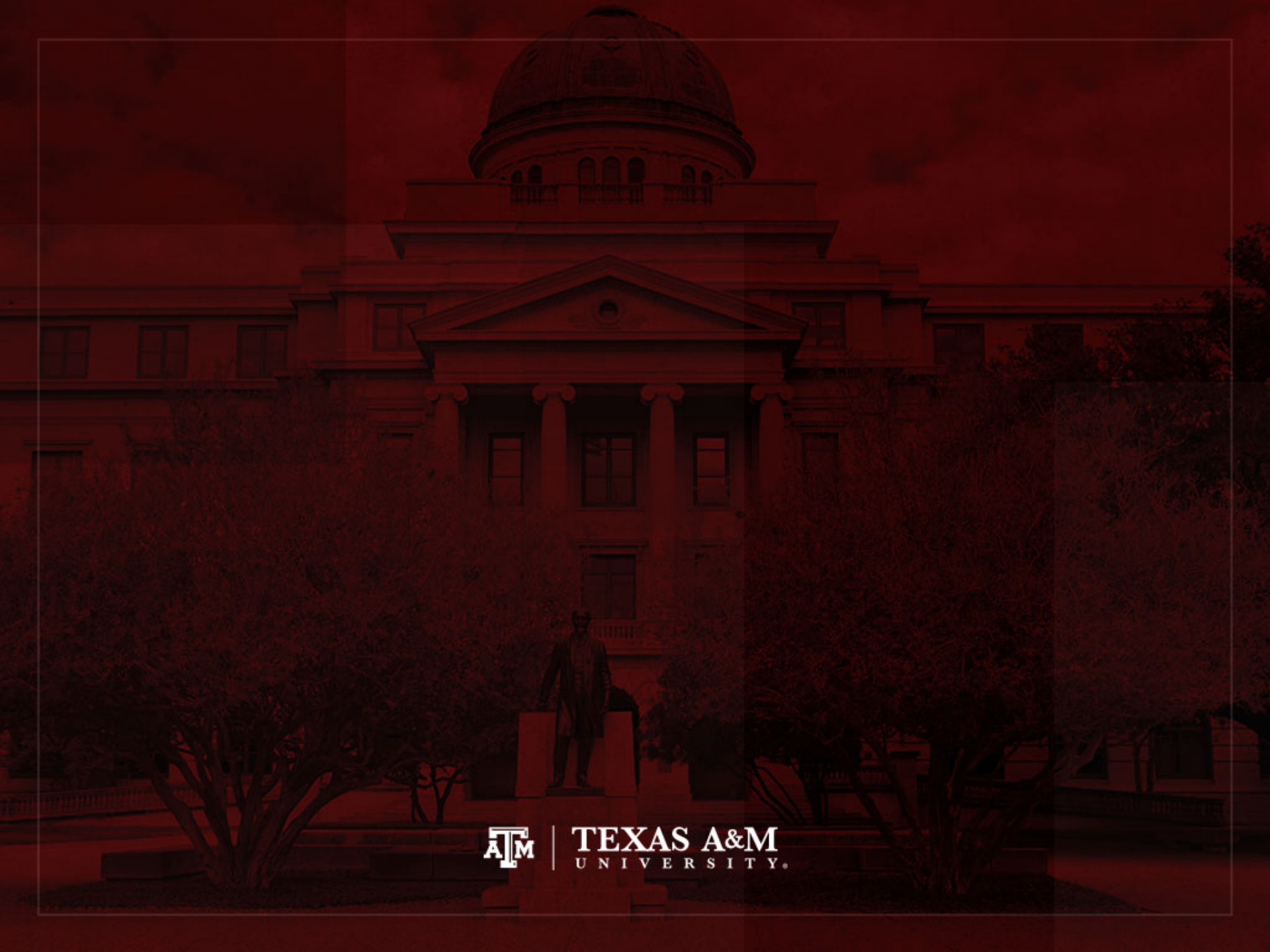
$${}_nL_{x+n} / {}_nL_x$$

- There is a different cohort for each age group x
 - L values for different columns of the Leslie matrix are L values from different cohort lifetables
 - They are L values from a period lifetable (chapter 7)

Period lifetable

- Period lifetable puts together survival experience for different cohorts as they move through the same time period
- The Leslie matrix does the same
- At this stage, the key concept is the way that ratios of big- L values supply elements for the subdiagonal matrix





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The Leslie matrix first row

- Outside the subdiagonal, the only elements in a Leslie matrix which are not structural zeros are in the first row
- These are the elements that take account of population renewal
 - Babies are born to parents
 - They survive and counted at the end of projection step
- Formulas for the first row are more complicated than those for the subdiagonal
 - Let's develop them one step at a time



Projecting female population

- Formulas are shown to project female population
 - Daughters are in first row, instead of sons and daughters
 - In projections, we must have the same kinds of people coming out as going in
- Joint projections for males and females are possible in principle
 - Simple and appealing two-sex model is a problem that remains largely unsolved



Projecting male population

- One-sex projections could be done for males
 - Inserting fertility rates for sons and fathers
 - But motherhood ages are more regular than fatherhood ages
- Usually the female population is projected
 - Counts of males are estimated from projected counts for females



Fraction female at birth

- When projecting females, we must remember that we need fertility rates for daughters only
- Published fertility rates are usually for babies of both sexes
- Need to multiply by the fraction female at birth (f_{fab})
 - By our default, it is the fraction 0.4886



Full formula for first row

- Formula for the expected number of daughters aged 0 to n at the end of the projection step
 - Per woman aged x to $x+n$ at the start
- We write $j(x)$ for the corresponding column with $x=j(x)(n)-n$

$$A_{1, j(x)} = \frac{{}_n L_0}{2 \ell_0} \left({}_n F_x + {}_n F_{x+n} \frac{{}_n L_{x+n}}{{}_n L_x} \right) f_{\text{fab}}$$

Formula step by step

- $A_{1, j(x)}$ pertains to
 - Daughters entering the population over n years
 - By being born to mothers aged x to $x+n$ at the start

Potential mothers at the start (x to $x+n$)

**Daughters
at the end
(0 to n)**

	0 to 5	5 to 10	10 to 15	15 to 20	20 to 25
0 to 5	kids	kids	kids	kids	kids
5 to 10	survivors	0	0	0	0
10 to 15	0	survivors	0	0	0
15 to 20	0	0	survivors	0	0
20 to 25	0	0	0	survivors	0

Crude version of formula

- How many daughters per potential mother should there be?
- A first guess would be to multiply
 - The daughters-only age-specific fertility rate $({}_nF_x)(f_{fab})$ for women x to $x+n$
 - By the years at risk in the interval (n)
- ${}_nF_x$ indicates period age-specific fertility rate
 - Instead of small ${}_nf_x$ for cohort age-specific fertility rate
- It provides a first crude version of the formula

$$(n)({}_nF_x)(f_{fab})$$

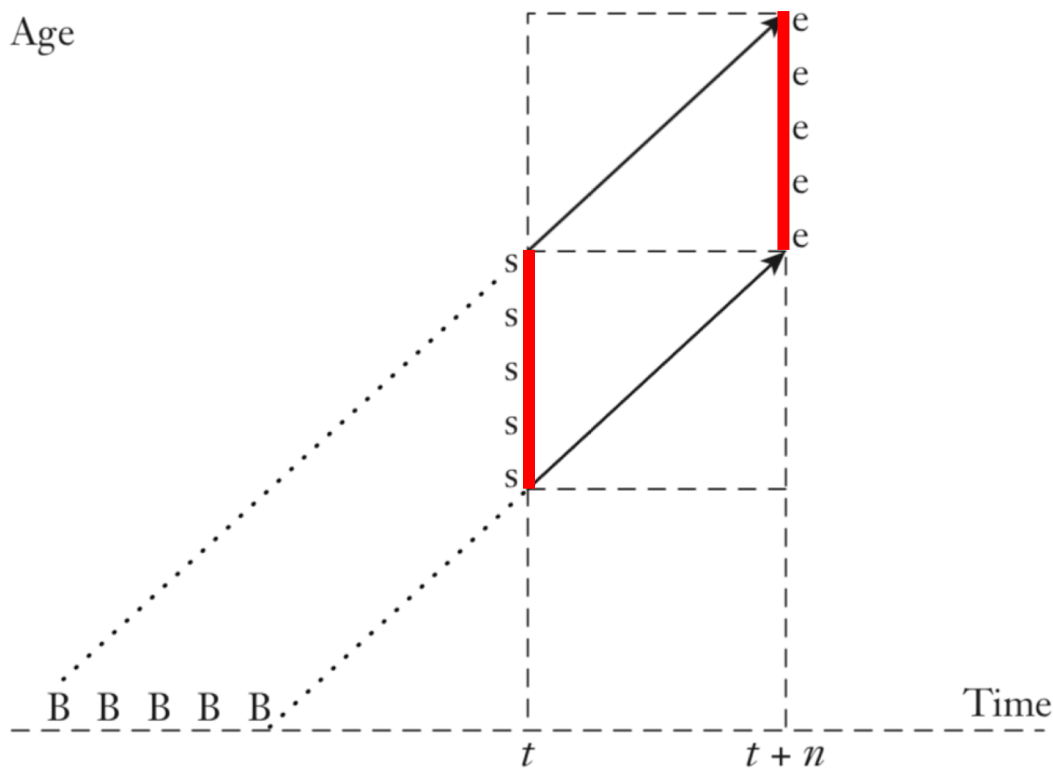


Consider mortality of daughters

- Need to recognize that not all baby daughters survive to the end of the projection interval
 - The first age group counts kids aged 0 to n at the end of the step, not newborns
 - Need to estimate proportion of babies born during the n years who survive to be counted
 - Babies born early in the period have to survive to be nearly n years old
 - Babies born late in the period have to survive only a little while

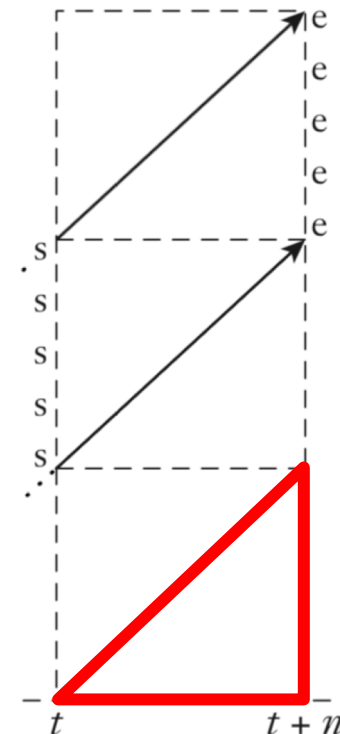
Survivorship for subdiagonal

- We averaged survivorships when we were finding a formula for subdiagonal elements of mothers
 - Compare lifelines crossing **two sides** of parallelogram



Survivorship for newborns

- We need average of survivorships for daughters
 - They start at birth, bottom axis of the Lexis diagram
 - Compare lifelines crossing two adjacent sides of a **right triangle**
 - Triangle base covers the interval from t to $t+n$
 - Perpendicular side reaches up from age 0 to age n above time $t+n$



Average of survivor daughters

- We ignore changes in initial cohort size within the interval
 - Out of any l_0 girls born near the start of the interval, about l_n survive to the end
 - Out of any l_0 born close to the end, nearly all l_0 survive to the end
- From $(n)(l_0)$ births, we expect this average of daughters who survive
 - $(n)(l_n + l_0) / 2$
 - This is a standard approximation for ${}_nL_0$



Better version of formula

- Average of daughters who survive: ${}_nL_0 = (n)(l_n + l_0)/2$
- Total births: $(n)(l_0)$
- Ratio of survivors to births: ${}_nL_0 / (n)(l_0)$
- Multiply this ratio by the first crude formula

$$\left(\frac{{}_nL_0}{(n)(l_0)} \right) (n)({}_nF_x)(f_{fab})$$

Mortality of daughters

Crude formula

n terms will cancel out

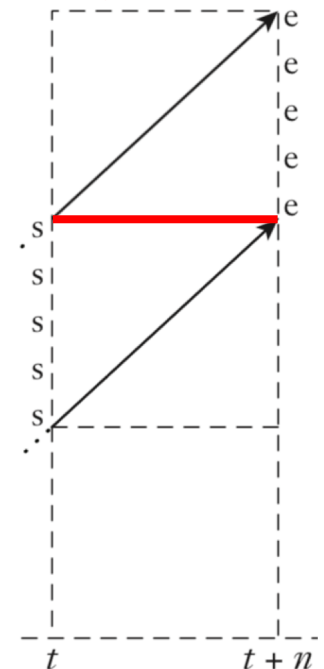
Consider aging of mothers

- Consider the aging of potential mothers during the projection interval
 - Women aged x to $x+n$ at the start only spend on average about half of the next n years in their starting age group
 - They grow older and spend about half the interval in the next age group
- In place of n years at ${}_nF_x: (n)({}_nF_x)$
 - We have about $n/2$ years at ${}_nF_x: (n/2)({}_nF_x)$
 - We have about $n/2$ years at ${}_nF_{x+n}: (n/2)({}_nF_{x+n})$



Consider mortality of mothers

- $n/2$ and $n/2$ are not quite the right breakdown
 - Not all women survive into the next age group
- We pretend that all deaths between start (s) and end (e) happen at the **age-group boundary** at age $x+n$
 - Then fraction of women surviving into the next age group is ${}_nL_{x+n}$ divided by ${}_nL_x$
 - It reduces time spent in the older age group (${}_nF_{x+n}$) by this survival fraction (${}_nL_{x+n} / {}_nL_x$)

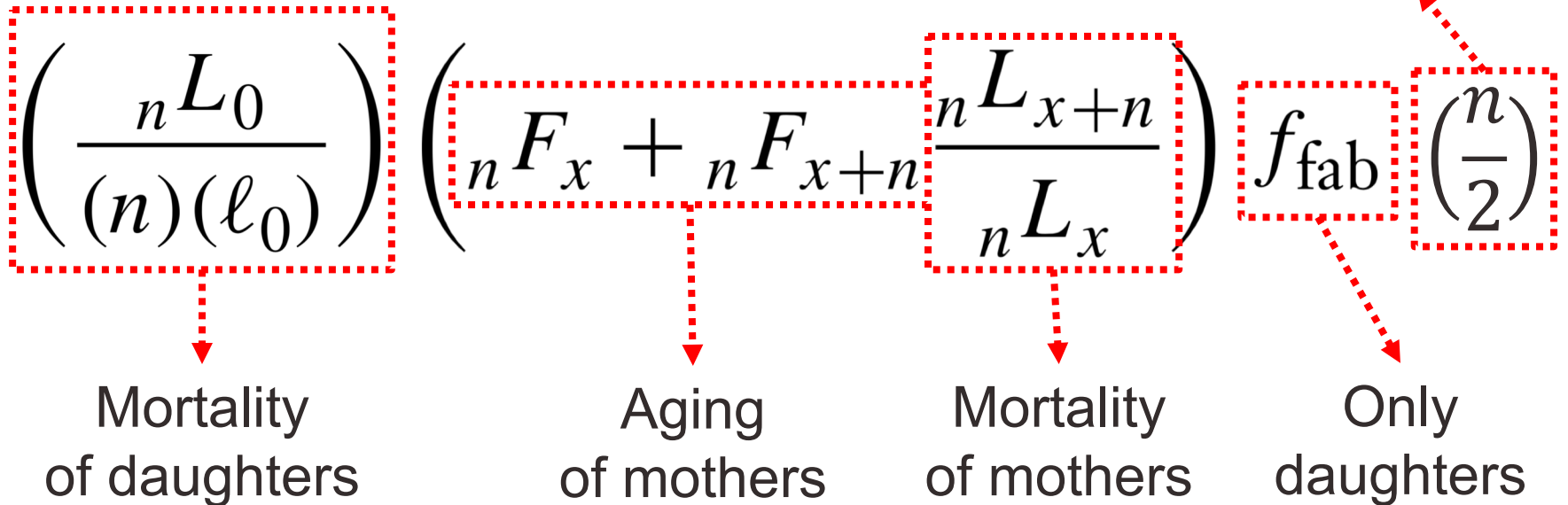


Aging and mortality of mothers

- Consider aging of mothers
 - Years spent in starting and ending age groups ($n/2$)
- Consider mortality of mothers
 - Reduce time spent in the older age group (${}_nF_{x+n}$) by this survival fraction (${}_nL_{x+n} / {}_nL_x$)

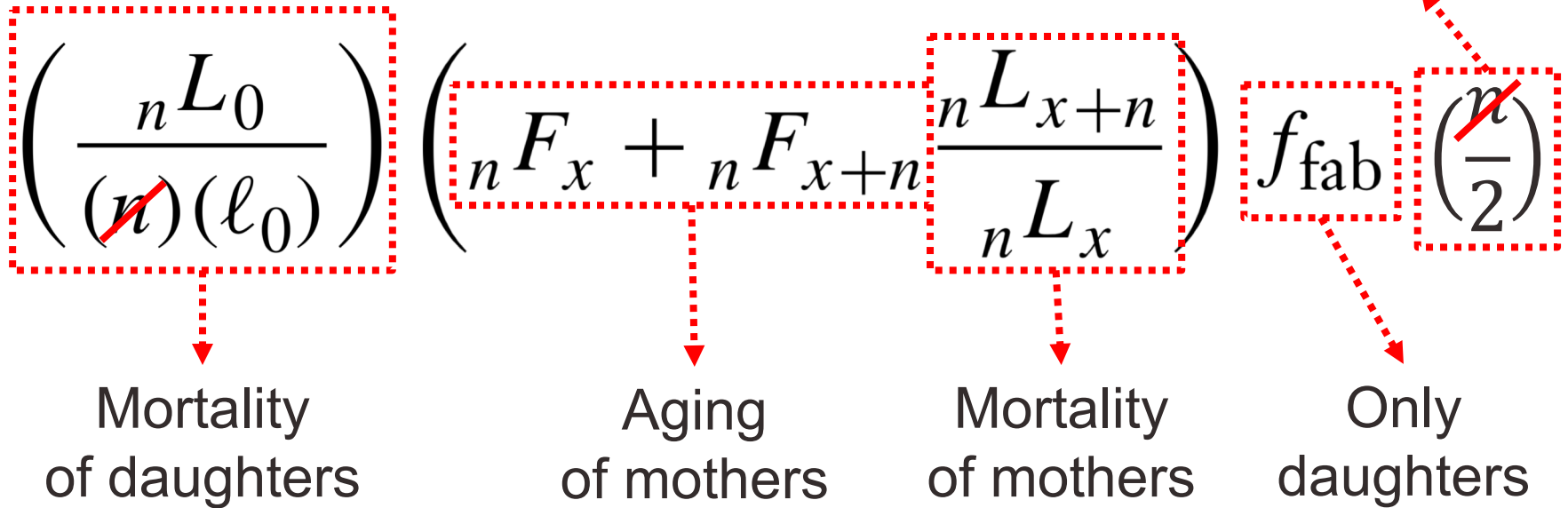
All elements of formula

Women live half of the time
in starting and next age groups



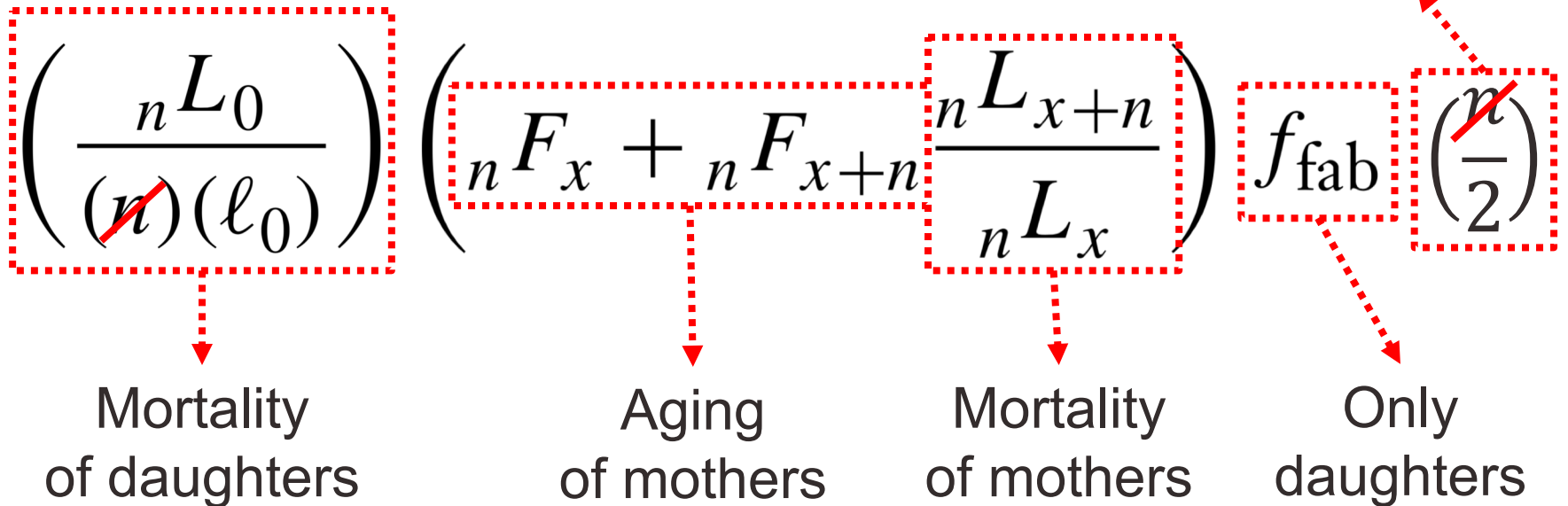
All elements of formula

Women live half of the time
in starting and next age groups



All elements of formula

Women live half of the time
in starting and next age groups



$$A_{1,j}(x) = \frac{{}_n L_0}{2 \ell_0} \left({}_n F_x + {}_n F_{x+n} \frac{{}_n L_{x+n}}{{}_n L_x} \right) f_{\text{fab}}$$

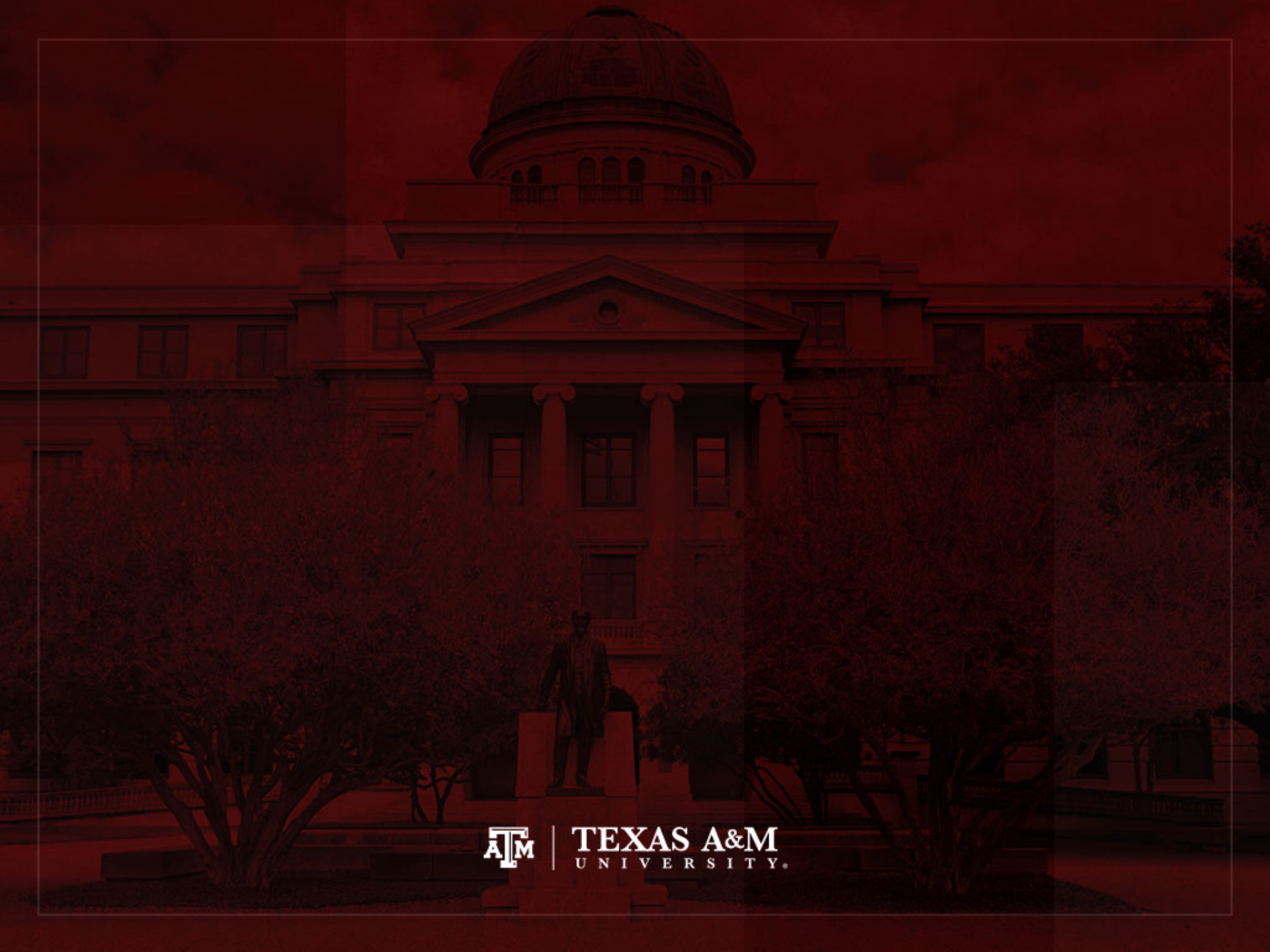
Fertility of youngest age group

- Usually the interval width (n) is 1 or 5 years
 - Age-specific fertility for the youngest age group from 0 to n will be zero
- The youngest age group is part of the “preprocreative span”
 - Period prior to procreation (production of infants)
- With wider intervals (10, 15, 20...), fertility for the youngest age group will not equal zero

Correction for youngest group

- With wider intervals (10, 15, 20...)
 - Children born during the projection step could grow up and bear their own children during one projection step
- Formulas made no allowance for grandchildren
 - A rough correction is to insert $(2)({}_nF_0)$, instead of ${}_nF_0$ for the upper-left element of the Leslie matrix
 - When fertility before age n is zero, this change has no effect
 - When it is not zero, the correction improves accuracy





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Projecting fillies, mares, seniors

- Example of a Leslie matrix and a population projection employing this matrix
 - No more than three rows and columns
 - Retain familiar 5-year age group
- Population of horses in a stable, with rates of survival and fertility that are stylized but credible
 - Fillies: young female horses (0–5)
 - Mares: mature females (5–10)
 - Seniors (10–15)



Age of reproduction

- Horses mature fairly quickly and may live to ages like 30
- In our stable they only give birth to offspring (foals) between ages 5–15
- We only project the population up to age 15
 - Leslie matrices often stop at the last age of reproduction
 - Sizes of older age groups can be computed directly from the relevant lifetable



Data for this example

- Age-specific fertility rates are

$${}_5F_0 f_{\text{fab}} = 0.000$$

$${}_5F_5 f_{\text{fab}} = 0.400$$

$${}_5F_{10} f_{\text{fab}} = 0.300$$

- Number of survivors to specific age

$$l_0 = 1.000$$

$$l_5 = 0.900$$

$$l_{10} = 0.600$$

$$l_{15} = 0.000$$



Three parts to complete

- We have three parts of the Leslie matrix to fill in
 - Structural zeros
 - Subdiagonal elements
 - First row

Structural zeros

- The first step in writing down the Leslie matrix is to fill in the structural zeros
 - Number 0 go on the structural zeros
- Crosses for nonzero elements go on
 - Subdiagonal
 - First row

$$\begin{pmatrix} X & X & X \\ X & 0 & 0 \\ 0 & X & 0 \end{pmatrix}$$

Subdiagonal

- Our second step is to compute survivorship ratios for the subdiagonal

$$A_{j+1, j} = \frac{{}_n L_{x+n}}{{}_n L_x} = \frac{(5/2)(l_{x+n} + l_{x+n+n})}{(5/2)(l_x + l_{x+n})}$$

$l_0 = 1.000$	Age group 0–5:	$A_{2\ 1} = \frac{(5/2)(0.9 + 0.6)}{(5/2)(1.0 + 0.9)} = 0.7895$
$l_5 = 0.900$		
$l_{10} = 0.600$	Age group 5–10:	$A_{3\ 2} = \frac{(5/2)(0.6 + 0.0)}{(5/2)(0.9 + 0.6)} = 0.4000$
$l_{15} = 0.000$		



First row

$$A_{1j} = \frac{{}_nL_0}{2 \ell_0} \left({}_nF_x + {}_nF_{x+n} \frac{{}_nL_{x+n}}{{}_nL_x} \right) (f_{\text{fab}})$$

$${}_5F_0 f_{\text{fab}} = 0.000$$

$${}_5L_0 = (5/2) (l_0 + l_5) = (5/2) (1 + 0.9) = 4.75$$

$${}_5F_5 f_{\text{fab}} = 0.400$$

$${}_5L_5 = (5/2) (l_5 + l_{10}) = (5/2) (0.9 + 0.6) = 3.75$$

$${}_5F_{10} f_{\text{fab}} = 0.300$$

$${}_5L_{10} = (5/2) (l_{10} + l_{15}) = (5/2) (0.6 + 0.0) = 1.50$$

$$\text{Age group 0–5: } A_{11} = \frac{4.75}{2} \left(0 + 0.400 \frac{3.75}{4.75} \right) = 0.7500$$

$$\text{Age group 5–10: } A_{12} = \frac{4.75}{2} \left(0.400 + 0.300 \frac{1.50}{3.75} \right) = 1.2350$$

$$\text{Age group 10–15: } A_{13} = \frac{4.75}{2} (0.300 + 0.000) = 0.7125$$



Leslie matrix for this example

$$\begin{pmatrix} 0.7500 & 1.2350 & 0.7125 \\ 0.7895 & 0 & 0 \\ 0 & 0.4000 & 0 \end{pmatrix}$$

Use matrix for projection

- Suppose at time $t=0$, we have
 - Zero fillies (0–5)
 - Four mares (5–10)
 - Two seniors (10–15)
- How many fillies, mares, and seniors should we expect after 5 years?

Rule for matrix multiplication

- If we write our population counts at time t as a vector $K(t)$, then we apply the standard rule for matrix multiplication
 - Expected population at time t (vector $K(t)$) equals the product of matrix A times vector $K(0)$
- The general rule for matrix products says that

$K(t) = A K(0)$, which means...

$$K_i(t) = \sum A_{i j} K_j(0)$$

- Summing over columns j of the matrix and elements j of the vector



How many fillies (0–5)?

- First row informs number of fillies to expect at the end of the 5-year step per horse at the start

$$\begin{pmatrix} 0.7500 & 1.2350 & 0.7125 \\ 0.7895 & 0 & 0 \\ 0 & 0.4000 & 0 \end{pmatrix}$$

(0.7500 fillies per starting filly) * (0 starting fillies)

+ (1.2350 fillies per starting mare) * (4 starting mares)

+ (0.7125 fillies per starting senior) * (2 starting seniors)

$$0 + 4.94 + 1.425 = 6.365 \text{ fillies}$$

- We should expect six or seven fillies
 - Around 1/3 chance of having 7 horses
 - Around 2/3 chance of having 6 horses



How many mares (5–10)?

- Use the second row of our matrix
- We go across the columns as we go down the starting vector $K(0)$
 - Zero starting fillies, four starting mares, two starting seniors
- We multiply and add up

$$\begin{pmatrix} 0.7500 & 1.2350 & 0.7125 \\ 0.7895 & 0 & 0 \\ 0 & 0.4000 & 0 \end{pmatrix}$$

$(0.7895 \text{ mares per starting filly}) * (0 \text{ starting fillies}) + 0 + 0$

0 mares



How many seniors (10–15)?

- Use the third row of our matrix
- We go across the columns as we go down the starting vector $K(0)$
 - Zero starting fillies, four starting mares, two starting seniors
- We multiply and add up

$$\begin{pmatrix} 0.7500 & 1.2350 & 0.7125 \\ 0.7895 & 0 & 0 \\ 0 & 0.4000 & 0 \end{pmatrix}$$

$$0 + (0.400 \text{ seniors per starting mare}) * (4 \text{ starting mares}) + 0$$

1.6 seniors



Summary

- Equation for population projection

$$K(t) = A K(0)$$

- Matrices

$$K(t) \quad \text{Leslie matrix} = \text{matrix } A \quad K(0)$$
$$\begin{pmatrix} 6.365 \\ 0.000 \\ 1.600 \end{pmatrix} = \begin{pmatrix} 0.7500 & 1.2350 & 0.7125 \\ 0.7895 & 0 & 0 \\ 0 & 0.4000 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix}$$



Matrix notation

- Matrix notation makes it easy to see what happens next

$$K(10) = AK(5) = AAK(0)$$

- $AA = A^2$ is not ordinary multiplication but matrix multiplication
- In general

$$K(nk) = A^k K(0)$$



Comparing to previous equation

- Equation from crude model of exponential growth

$$K(T) = A^T K(0)$$

- It has ordinary numbers rather than vector
- It does not account for age

- New equation for population projection

$$K(nk) = A^k K(0)$$

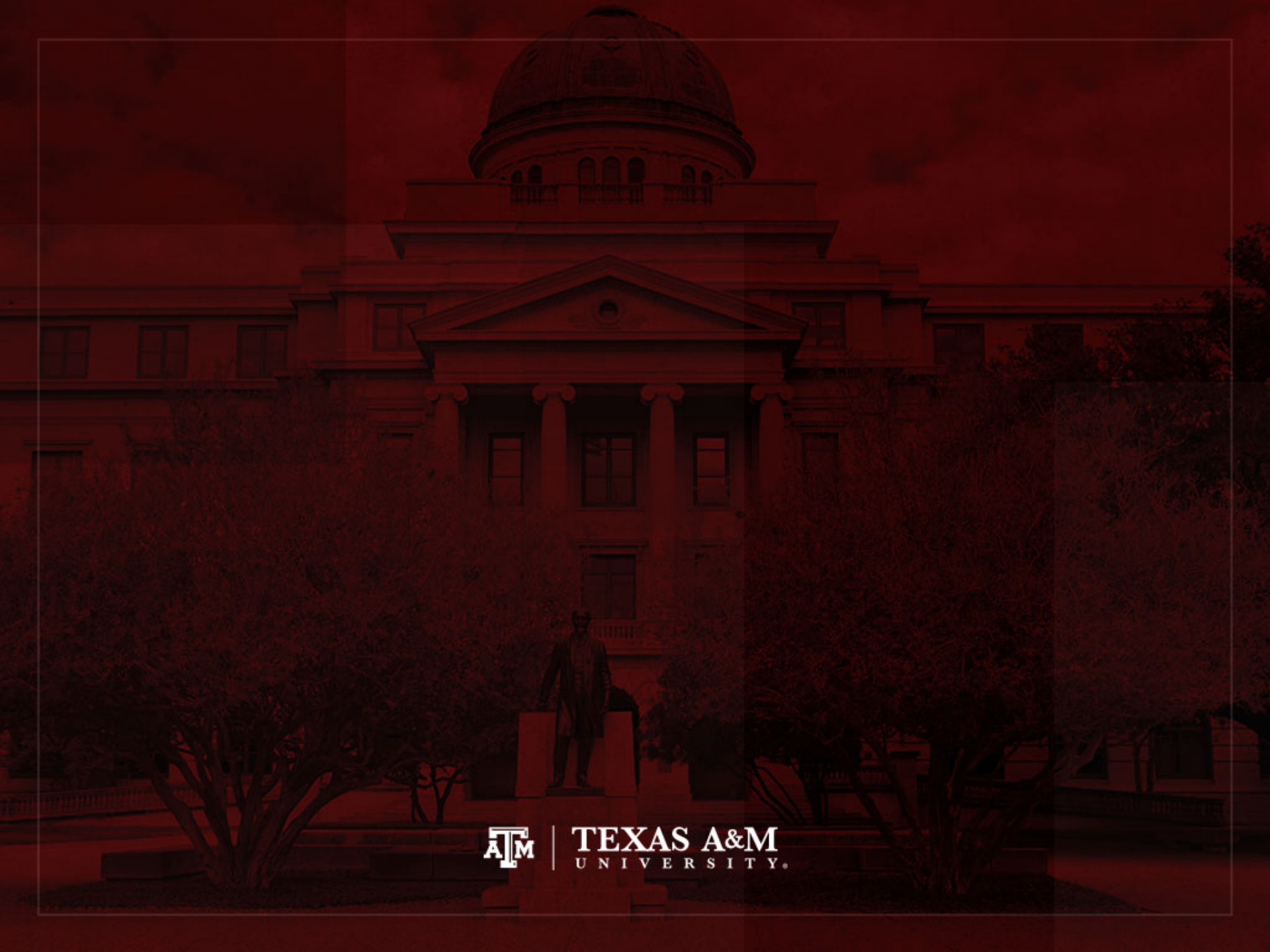
- It utilizes matrices and vectors, which consider age
- Same form as before, but new and richer interpretation
- This is the generalization to age-structured populations of the Crude Rate Model, but it is still a closed population (no migration)



References

Wachter KW. 2014. Essential Demographic Methods. Cambridge: Harvard University Press. Chapter 5 (pp. 98–124).





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