

# Lecture 10: Analysis of variance

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Source: Healey, Joseph F. 2015. "Statistics: A Tool for Social Research." Stamford: Cengage Learning. 10th edition. Chapter 10 (pp. 247–275).



# Outline

- Identify and cite examples of situations in which analysis of variance (ANOVA) is appropriate
- Explain the logic of hypothesis testing as applied to ANOVA
- Perform the ANOVA test, using the five-step model as a guide, and correctly interpret the results
- Define and explain the concepts of population variance, total sum of squares, sum of squares between, sum of squares within, mean square estimates
- Explain the difference between the statistical significance and the importance (magnitude) of relationships between variables



# ANOVA application

- ANOVA can be used in situations where the researcher is interested in the differences in sample means across three or more categories
  - How do Protestants, Catholics, and Jews vary in terms of number of children?
  - How do Republicans, Democrats, and Independents vary in terms of income?
  - How do older, middle-aged, and younger people vary in terms of frequency of church attendance?

# Extension of $t$ -test

- We can think of ANOVA as an extension of  $t$ -test for more than two groups
  - Are the differences between the samples large enough to reject the null hypothesis and justify the conclusion that the populations represented by the samples are different?
- Null hypothesis,  $H_0$ 
  - $H_0: \mu_1 = \mu_2 = \mu_3 = \dots = \mu_k$
  - All population means are similar to each other
- Alternative hypothesis,  $H_1$ 
  - At least one of the populations means is different



# Logic of ANOVA

- Could there be a relationship between age and support for capital punishment?
  - No difference between groups

**Support for Capital Punishment by Age Group (fictitious data)**

	18–29	30–45	46–64	65+
Mean	10.3	11.0	10.1	9.9
Standard deviation	2.4	1.9	2.2	1.7

- Difference between groups

**Support for Capital Punishment by Age Group (fictitious data)**

	18–29	30–45	46–64	65+
Mean	10.0	13.0	16.0	22.0
Standard deviation	2.4	1.9	2.2	1.7



# Between and within differences

- If the  $H_0$  is true, the sample means should be about the same value
  - If the  $H_0$  is true, there will be little difference between sample means
- If the  $H_0$  is false
  - There should be substantial differences **between** sample means (between categories)
  - There should be relatively little difference **within** categories
    - The sample standard deviations should be small within groups



# Likelihood of rejecting $H_0$

- The greater the difference between categories (as measured by the means)
  - Relative to the differences within categories (as measured by the standard deviations)
  - The more likely the  $H_0$  can be rejected
- When we reject  $H_0$ 
  - We are saying there are differences between the populations represented by the sample



# Computation of ANOVA

1. Find total sum of squares ( $SST$ )

$$SST = \sum (X_i^2) - n\bar{X}^2$$

2. Find sum of squares between ( $SSB$ )

$$SSB = \sum [n_k (\bar{X}_k - \bar{X})^2]$$

- $SSB$  = sum of squares between categories
- $n_k$  = number of cases in a category
- $\bar{X}_k$  = mean of a category

3. Find sum of squares within ( $SSW$ )

$$SSW = SST - SSB$$





# 4. Degrees of freedom

$$dfw = n - k$$

- $dfw$  = degrees of freedom within
- $n$  = total number of cases
- $k$  = number of categories

$$dfb = k - 1$$

- $dfb$  = degrees of freedom between
- $k$  = number of categories



# Final estimations

5. Find mean square estimates

$$\text{Mean square within} = \frac{SSW}{dfw}$$

$$\text{Mean square between} = \frac{SSB}{dfb}$$

6. Find the  $F$  ratio

$$F(\text{obtained}) = \frac{\text{Mean square between}}{\text{Mean square within}}$$



# Example

- Support for capital punishment
- Sample of 16 people who are equally divided across four age groups

Support for Capital Punishment by Age Group (fictitious data)

18–29		30–45		46–64		65+	
$X_i$	$X_i^2$	$X_i$	$X_i^2$	$X_i$	$X_i^2$	$X_i$	$X_i^2$
7	49	10	100	12	144	17	289
8	64	12	144	15	225	20	400
10	100	13	169	17	289	24	576
15	225	17	289	20	400	27	729
<u>40</u>	<u>438</u>	<u>52</u>	<u>702</u>	<u>64</u>	<u>1058</u>	<u>88</u>	<u>1994</u>
$\bar{X}_k = 10.0$		$\bar{X}_k = 13.0$		$\bar{X}_k = 16.0$		$\bar{X}_k = 22.0$	
$\bar{X} = 15.25$							



# Step 1: Assumptions, requirements

- Independent random samples
- Interval-ratio level of measurement
- Normally distributed populations
- Equal population variances



# Step 2: Null hypothesis

- Null hypothesis,  $H_0: \mu_1 = \mu_2 = \mu_3 = \mu_4$ 
  - The null hypothesis asserts there is no difference between the populations
- Alternative hypothesis,  $H_1$ 
  - At least one of the populations means is different



# Step 3: Distribution, critical region

- Sampling distribution
  - $F$  distribution
- Significance level
  - Alpha ( $\alpha$ ) = 0.05
- Degrees of freedom
  - $dfw = n - k = 16 - 4 = 12$
  - $dfb = k - 1 = 4 - 1 = 3$
- Critical  $F$ 
  - $F(\text{critical}) = 3.49$





# Step 4: Test statistic

1. Total sum of squares ( $SST$ )

$$SST = \sum (X_i^2) - n\bar{X}^2$$

$$SST = (438 + 702 + 1058 + 1994) - (16)(15.25)^2$$

$$SST = 471.04$$

2. Sum of squares between ( $SSB$ )

$$SSB = \sum [n_k(\bar{X}_k - \bar{X})^2]$$

$$SSB = 4(10 - 15.25)^2 + 4(13 - 15.25)^2 \\ + 4(16 - 15.25)^2 + 4(22 - 15.25)^2 = 314.96$$

3. Sum of squares within ( $SSW$ )

$$SSW = SST - SSB = 471.04 - 314.96 = 156.08$$



#### 4. Degrees of freedom

$$dfw = n - k = 16 - 4 = 12$$

$$dfb = k - 1 = 4 - 1 = 3$$

#### 5. Mean square estimates

$$\text{Mean square within} = \frac{SSW}{dfw} = \frac{156.08}{12} = 13.00$$

$$\text{Mean square between} = \frac{SSB}{dfb} = \frac{314.96}{3} = 104.99$$

#### 6. $F$ ratio

$$F(\text{obtained}) = \frac{\text{Mean square between}}{\text{Mean square within}} = \frac{104.99}{13.00} = 8.08$$



# Step 5: Decision, interpret

- $F(\text{obtained}) = 8.08$
- This is beyond  $F(\text{critical}) = 3.49$
- The obtained test statistic falls in the critical region, so we **reject** the  $H_0$
- Support for capital punishment does differ across age groups



# Limitations of ANOVA

- Requires interval-ratio level measurement of the dependent variable
- Requires roughly equal numbers of cases in the categories of the independent variable
- Statistically significant differences are not necessarily important (small magnitude)
- The alternative (research) hypothesis is not specific
  - It only asserts that at least one of the population means differs from the others



# Example from 2016 GSS

- We know the average income by race/ethnicity

```
. tabstat conrinc [aweight=wtssall], by(raceeth) stat(mean sd n)
```

```
Summary for variables: conrinc  
Group variable: raceeth (Race/Ethnicity)
```

raceeth	Mean	SD	N
White	38845.62	39157.17	1072
Black	23243.04	19671.53	273
Hispanic	23128.92	21406.31	215
Other	50156.35	59219.9	72
Total	34649.3	36722.06	1632

- Does at least one category of the race/ethnicity variable have average income different than the others?
  - This is not a perfect example for ANOVA, because the race/ethnicity variable does not have equal numbers of cases across its categories



# Example from GSS: Result

- The probability of not rejecting  $H_0$  is small ( $p < 0.01$ )
  - At least one category of the race/ethnicity variable has average income different than the others with a 99% confidence level
  - However, ANOVA does not inform which category has an average income significantly different than the others in 2016

```
. oneway conrinc raceeth [aweight=wtssall]
```

Source	Analysis of variance				Prob > F
	SS	df	MS	F	
Between groups	<b>1.0142e+11</b>	<b>3</b>	<b>3.3806e+10</b>	<b>26.23</b>	<b>0.0000</b>
Within groups	<b>2.0980e+12</b>	<b>1628</b>	<b>1.2887e+09</b>		
Total	<b>2.1994e+12</b>	<b>1631</b>	<b>1.3485e+09</b>		

Bartlett's equal-variances test:  $\chi^2(3) = 292.7013$       Prob> $\chi^2 = 0.000$

Source: 2016 General Social Survey.





# Edited table

**Table 1. One-way analysis of variance for individual average income of the U.S. adult population by race/ethnicity, 2004, 2010, and 2016**

Source	Sum of Squares	Degrees of Freedom	Mean of Squares	F-test	Prob > F
<b>2004</b>					
Between groups	5.92e+10	3	1.97e+10	16.36	0.0000
Within groups	2.03e+12	1,682	1.21e+09		
Total	2.09e+12	1,685	1.24e+09		
<b>2010</b>					
Between groups	6.02e+10	3	2.01e+10	24.50	0.0000
Within groups	9.79e+11	1,195	819,590,864		
Total	1.04e+12	1,198	867,818,893		
<b>2016</b>					
Between groups	1.01e+11	3	3.38e+10	26.23	0.0000
Within groups	2.10e+12	1,628	1.29e+09		
Total	2.20e+12	1,631	1.35e+09		

Source: 2004, 2010, 2016 General Social Surveys.



# Example from 2019 ACS, Texas

- We know the average income by race/ethnicity

```
. tabstat income if income!=0 & income!=. [fweight=perwt], by(raceth) stat(mean sd n)
```

Summary for variables: income  
Group variable: raceth

raceth	Mean	SD	N
White	63199.24	74601.04	6081513
African American	40079.03	40410.99	1766063
Hispanic	36595.08	38076.88	5250789
Asian	66528.88	73827.69	776722
Native American	44246.01	57666.53	44743
Other races	46151.98	58649.93	235029
Total	50285.44	60567.56	1.42e+07

- Does at least one category of race/ethnicity have average income different than the others?
  - This is not a perfect example for ANOVA, because race/ethnicity does not have equal numbers of cases across its categories

```
. svy, subpop(if income!=0 & income!=.): mean income, over(raceth)
(running mean on estimation sample)
```

```
. estat sd
(correct standard deviation)
```

Over	Mean	Std. dev.
c.income@ raceth		
White	63199.24	81952.97
African A..	40079.03	33729.03
Hispanic	36595.08	34417.96
Asian	66528.88	71633.26
Native Am..	44246.01	57876.89
Other races	46151.98	56501.55

```
. svy, subpop(if income!=0 & income!=.): mean income
(running mean on estimation sample)
```

```
. estat sd
```

	Mean	Std. dev.
income	50285.44	59920.72

# Example from ACS: Result

- The probability of not rejecting  $H_0$  is small ( $p < 0.01$ )
  - At least one category of the race/ethnicity variable has average income different than the others with a 99% confidence level
  - However, ANOVA does not inform which category has an average income significantly different than the others

`. oneway income raceth if income!=0 & income!=. [aweight=perwt]`

Analysis of variance					
Source	SS	df	MS	F	Prob > F
Between groups	2.2032e+13	5	4.4065e+12	1259.17	<b>0.0000</b>
Within groups	4.5608e+14	130325	3.4995e+09		(statistical significance)
Total	4.7811e+14	130330	3.6685e+09		

Bartlett's equal-variances test:  $\chi^2(5) = 1.2e+04$  Prob> $\chi^2 = 0.000$

Source: 2019 American Community Survey, Texas.



# Example from 2019 ACS: n, N

```
. ***Sample size of each category of race/ethnicity and missing cases
. tab raceth if income!=0 & income!=., m
```

raceth	Freq.	Percent	Cum.
White	69,043	52.98	52.98
African American	11,574	8.88	61.86
Hispanic	40,359	30.97	92.82
Asian	6,879	5.28	98.10
Native American	424	0.33	98.43
Other races	2,052	1.57	100.00
Total	130,331	100.00	

```
. ***Population size of each category of race/ethnicity
. tab raceth if income!=0 & income!=. [fweight=perwt]
```

raceth	Freq.	Percent	Cum.
White	6,081,513	42.96	42.96
African American	1,766,063	12.48	55.44
Hispanic	5,250,789	37.10	92.54
Asian	776,722	5.49	98.02
Native American	44,743	0.32	98.34
Other races	235,029	1.66	100.00
Total	14,154,859	100.00	

(correct percentage distribution)



# Edited table

**Table 1. One-way analysis of variance for wage and salary income by race/ethnicity, Texas, 2019**

Race/ethnicity	Income		Population percentage
	Mean	Standard deviation	
White	63,199.24	81,952.97	42.96
African American	40,079.03	33,729.03	12.48
Hispanic	36,595.08	34,417.96	37.10
Asian	66,528.88	71,633.26	5.49
Native American	44,246.01	57,876.89	0.32
Other races	46,151.98	56,501.55	1.66
Total	50,285.44	59,920.72	100.00
Population size	—	—	14,154,859
Sample size	—	—	130,331

  

ANOVA	Sum of squares	Degrees of freedom	Mean of squares	F-test	Prob > F
Between groups	2.20e+13	5	4.41e+12	1,259.17	0.0000
Within groups	4.56e+14	130,325	3.50e+09		
Total	4.78e+14	130,330	3.67e+09		







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