Lecture 8: Hypothesis testing: One-sample case

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Advanced Methods of Social Research (SOCI 420)

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Source: Healey, Joseph F. 2015. "Statistics: A Tool for Social Research." Stamford: Cengage Learning. 10th edition. Chapter 8 (pp. 185–215).

TEXAS A&M UNIVERSITY

Outline

- Explain the logic of hypothesis testing, including concepts of the null hypothesis, the sampling distribution, the alpha level, and the test statistic
- Explain what it means to “reject the null hypothesis” or “do not reject the null hypothesis”
- Identify and cite examples of situations in which one-sample tests of hypotheses are appropriate
- Test the significance of single-sample means and proportions using the five-step model, and correctly interpret the results
- Explain the difference between one- and two-tailed tests, and specify when each is appropriate
- Define and explain Type I and Type II errors, and relate each to the selection of an alpha level
- Use the Student’s *t* distribution to test the significance of a sample mean for a small sample

Significant differences

- Hypothesis testing is designed to detect significant differences
  - Differences that did not occur by random chance
  - Hypothesis testing is also called significance testing
- This chapter focuses on the “one sample” case
  - Compare a random sample against a population
  - Compare a sample statistic to a (hypothesized) population parameter to see if there is a statistically significant difference

Example 1: Question

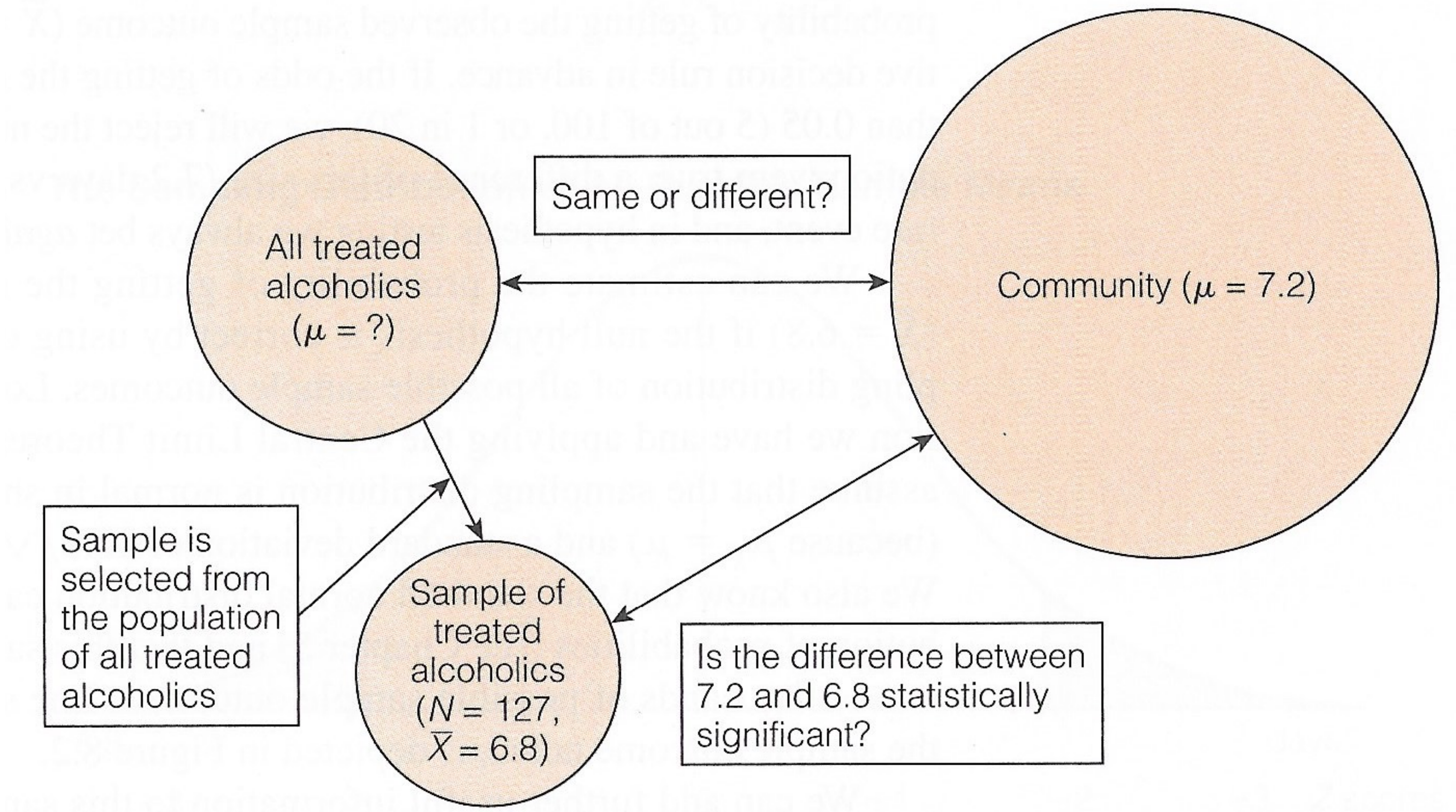
- Are people who have been treated for alcoholism more reliable workers than those in the community?
  - Does the group of all treated alcoholics have different absentee rates than the community as a whole?
  - Effectiveness of rehabilitation center for alcoholics
- Absentee rates for community and sample
  - Don't have resources to gather information of all people who have been treated by the program

Community	Sample of treated alcoholics
$\mu = 7.2\ days\ per\ year$	$\bar{X} = 6.8\ days\ per\ year$
$\sigma = 1.43$	$n = 127$

- What causes the difference between 7.2 and 6.8?
  - Real difference? Or difference due to random chance?

Source: Healey 2015, p.187.

A test of hypothesis for single-sample means

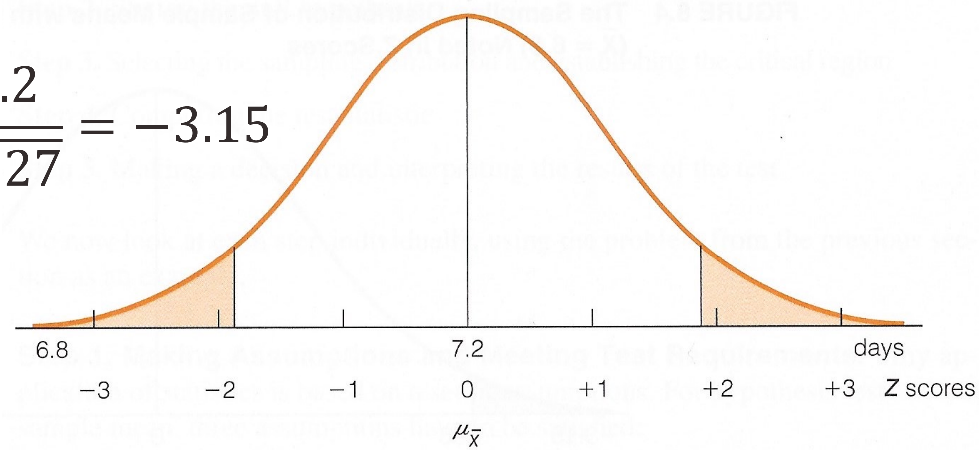


Source: Healey 2015, p.187.

Example 1: Result

- For a known/empirical distribution, we use: $Z = \frac{X_i - \bar{X}}{s}$
- However, we are concerned with the sampling distribution of all possible sample means

$$Z(obtained) = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{6.8 - 7.2}{1.43/\sqrt{127}} = -3.15$$



- The sample outcome falls in the shaded area
  - *Z(obtained)* = –3.15
  - Reject H_0: *μ* = 7.2 days per year
  - The sample of 127 treated alcoholics comes from a population that is significantly different from the community on absenteeism

Source: Healey 2015, p.189.

The five-step model

1. Make assumptions and meet test requirements
2. Define the null hypothesis (H_0)
3. Select the sampling distribution and establish the critical region
4. Compute the test statistic
5. Make a decision and interpret the test results

Example 2: Question

- The education department at a university has been accused of "grade inflation"
  - Thus, education majors have much higher GPAs than students in general
- GPAs of all education majors should be compared with the GPAs of all students
  - There are 1000s of education majors, far too many to interview
  - How can the dispute be investigated without interviewing all education majors?

Example 2: Numbers

- The average GPA for all students is 2.70 (μ)
  - This value is a parameter
- Random sample of education majors
  - Mean = $\bar{X}$ = 3.00
  - Standard deviation = s = 0.70
  - Sample size = n = 117
- There is a difference between parameter (μ=2.70) and statistic ($\bar{X}$=3.00)
  - It seems that education majors do have higher GPAs

Example 2: Explanations

- We are working with a random sample
  - Not all education majors

- Two explanations for the difference

1. The sample mean ($\bar{X}$=3.00) is the same as the population mean (μ=2.70)
   - The observed difference may have been caused by random chance

2. The difference is real (statistically significant)
   - Education majors are different from all students

Step 1: Assumptions,requirements

- Make assumptions
  - Random sampling
  - Hypothesis testing assumes samples were selected according to EPSEM
- Meet test requirements
  - The sample of 117 was randomly selected from all education majors
  - Level of measurement is interval-ratio
    - GPA is an interval-ratio level variable, so the mean is an appropriate statistic
  - Sampling distribution is normal in shape
    - This is a large sample ($n \geq 100$)

Step 2: Null hypothesis

- Null hypothesis, H_0: $\mu = 2.7$
  - H_0 always states there is no significant difference
  - The sample of 117 comes from a population that has a GPA of 2.7
  - The difference between 2.7 and 3.0 is trivial and caused by random chance
- Alternative hypothesis, H_1: $\mu \neq 2.7$
  - H_1 always contradicts H_0
  - The sample of 117 comes from a population that does not have a GPA of 2.7
  - There is an actual difference between education majors ($\bar{X}$=3.0) and other students (μ=2.7)

Step 3: Distribution, critical region

- Sampling distribution: standard normal shape
  - Alpha (α) = 0.05
  - Use the 0.05 value as a guideline to identify differences that would be rare if H_0 is true
  - Any difference with a probability less than α is rare and will cause us to reject the H_0
- Use the Z score to determine the probability of getting the observed difference
  - If the probability is less than 0.05, the obtained Z score will be beyond the critical Z score of ±1.96
  - This is the critical Z score associated with a two-tailed test and α=0.05

Step 4: Test statistic

- For a known/empirical distribution, we would use

$$Z = \frac{X_i - \bar{X}}{s}$$

- However, we are concerned with the sampling distribution of all sample means

- We only have the standard deviation for the sample (s), not for the population (σ)

$$Z(obtained) = \frac{\bar{X} - \mu}{s/\sqrt{n-1}} = \frac{3.0 - 2.7}{0.7/\sqrt{117-1}} = 4.62$$

Step 5: Decision, interpret

- *Z(obtained)* = 4.62
  - This is beyond *Z(critical)* = ±1.96
  - The obtained *Z* score fell in the critical region, so we ***reject*** the H_0
  - If H_0 was true, a sample GPA of 3.0 would be unlikely
  - Therefore, the H_0 is false and must be rejected
- Education majors have a GPA that is significantly higher than general student body
  - The difference between the parameter (μ=2.7) and the statistic ($\bar{X}$=3.0) was large and unlikely to have occurred by random chance (p<0.05)

Five-step model summary

Situation	Decision	Interpretation
The test statistic is in the critical region	Reject the null hypothesis (H_0)	The difference is statistically significant
The test statistic is not in the critical region	Do not reject the null hypothesis (H_0)	The difference is not statistically significant

- Model is fairly rigid, but there are two crucial choices
  - One-tailed or two-tailed test
  - Alpha (α) level

Source: Healey 2015, p.194.

One or two-tailed test

- Null hypothesis always has the equal sign

$$H_0\text{: } \mu = 2.7$$

- Two-tailed test states that population mean is not equal to the value stated in null hypothesis

$$H_1\text{: } \mu \neq 2.7$$

- One-tailed test estimates differences in a specific direction (based on theory)

$$H_1\text{: } \mu > 2.7$$

$$H_1\text{: } \mu < 2.7$$

One or two-tailed test

One- vs. Two-Tailed Tests, $\alpha = 0.05$

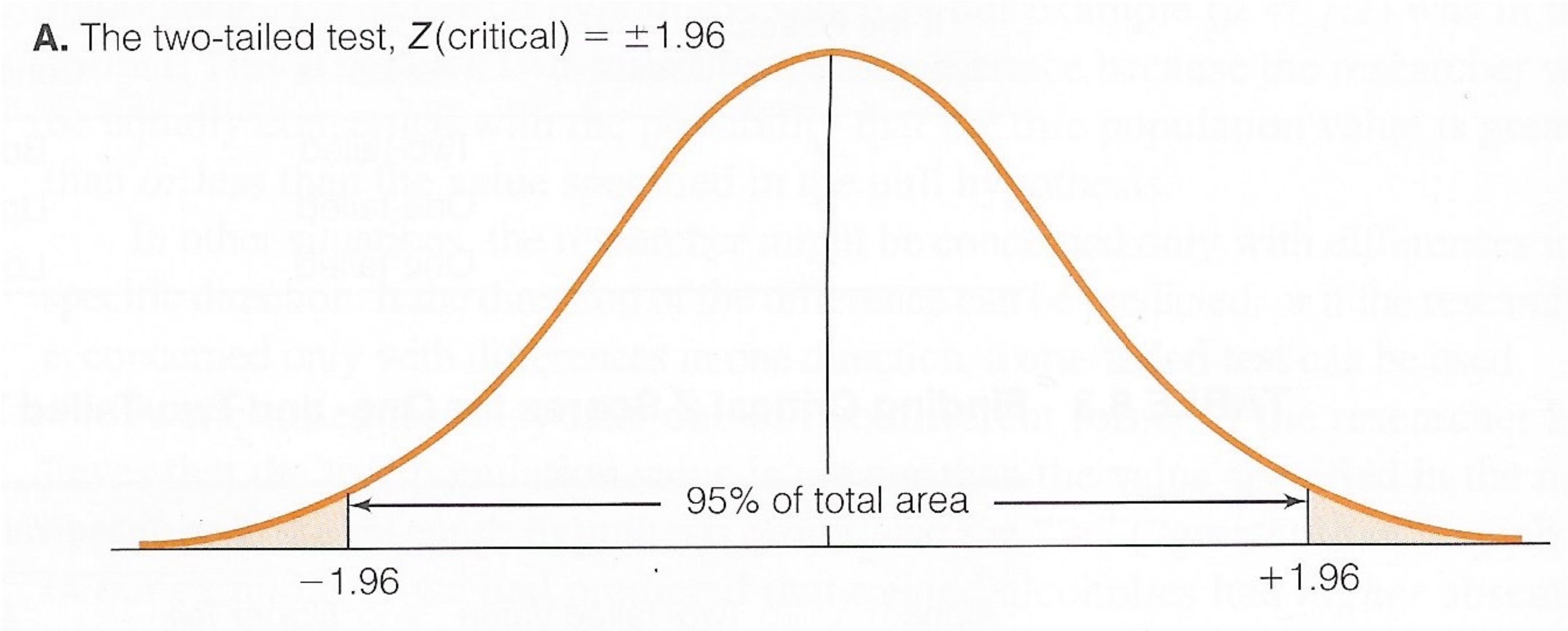
If the Research Hypothesis (H_1) Uses	The Test Is	Concern Is on	Z(critical) Is
$\neq$	Two-tailed	Both tails	± 1.96
$>$	One-tailed	Upper tail	$+1.65$
$<$	One-tailed	Lower tail	-1.65

Finding Critical Z Scores for One- and Two-Tailed Tests

		One-Tailed Value	
Alpha	Two-Tailed Value	*Upper Tail*	*Lower Tail*
0.10	± 1.65	$+1.29$	-1.29
0.05	± 1.96	$+1.65$	-1.65
0.01	± 2.58	$+2.33$	-2.33
0.001	± 3.32	$+3.10$	-3.10
0.0001	± 3.90	$+3.70$	-3.70

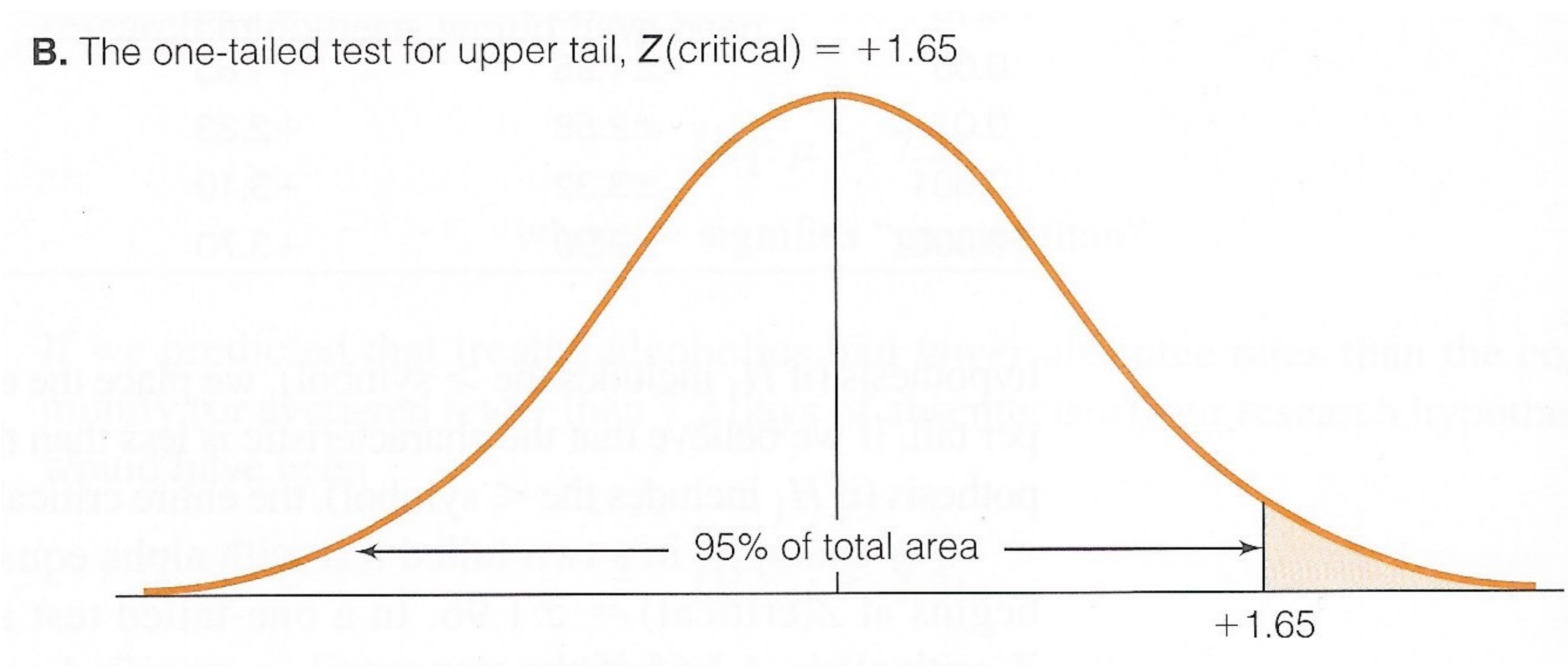
Source: Healey 2015, p.197.

Two-tailed test: α=0.05



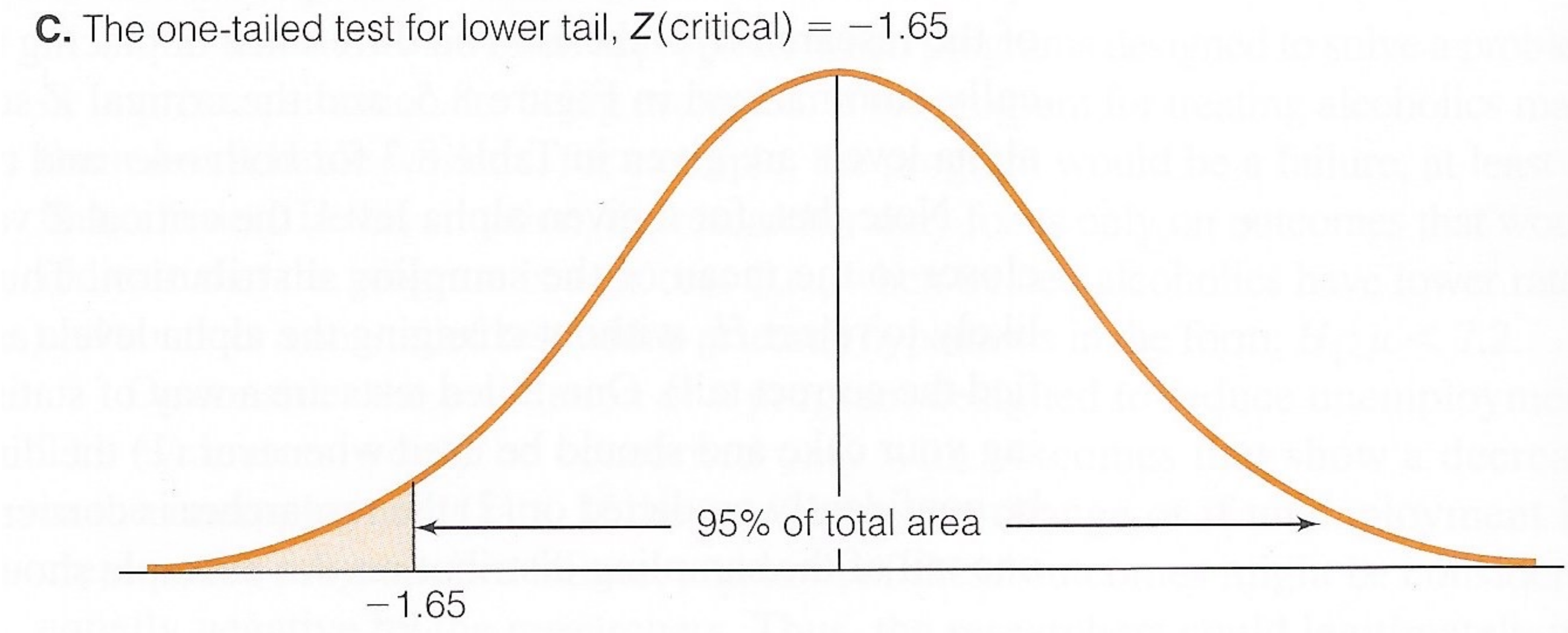
Source: Healey 2015, p.198.

One-tailed test (upper): α=0.05



Source: Healey 2015, p.198.

One-tailed test (lower): α=0.05



Source: Healey 2015, p.198.

Selecting an alpha level

- By assigning an alpha level, one defines an “unlikely” sample outcome
- Alpha level is the probability that the decision to reject the null hypothesis is incorrect
- Examine this table for critical regions

The Relationship Between Alpha and *Z*(Critical) for a Two-Tailed Test

If Alpha =	The Two-Tailed Critical Region Will Begin at *Z*(Critical) =
0.100	±1.65
0.050	±1.96
0.010	±2.58
0.001	±3.32

Source: Healey 2015, p.201.

Type I and Type II errors

- Type I error (alpha error)
  - Rejecting a true null hypothesis
- Type II error (beta error)
  - Not rejecting a false null hypothesis
- Examine table below for relationships between decision making and errors

Decision Making and the Five-Step Model

	If Our Decision Is to	And H_0 Is Actually	The Result Is
a	Reject H_0	False	OK
b	Fail to reject H_0	True	OK
c	Reject H_0	True	Type I or alpha (α) error
d	Fail to reject H_0	False	Type II or beta (β) error

Source: Healey 2015, p.201.

Decisions about hypotheses

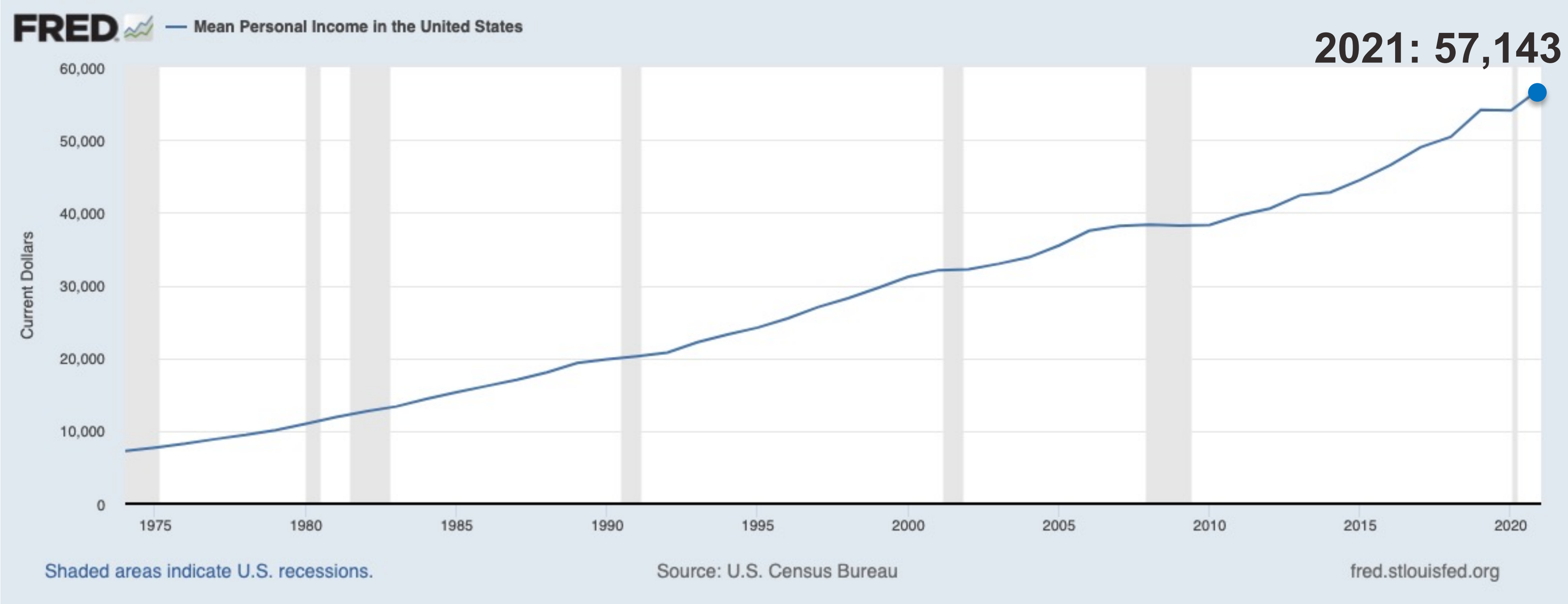
Hypotheses	$p < \alpha$	$p > \alpha$
Null hypothesis (H_0)	Reject	Do not reject
Alternative hypothesis (H_1)	Accept	Do not accept

- ***p*-value** is the probability of not rejecting the null hypothesis
- If a statistical software gives only the two-tailed *p*-value, divide it by 2 to obtain the one-tailed *p*-value

Significance level (α)	Confidence level
0.10 (10%)	90%
0.05 (5%)	95%
0.01 (1%)	99%
0.001 (0.1%)	99.9%

Example 3: Income, 2021

- Is the mean personal income of Veterans (GSS) lower than mean income of population 15+ (Census Bureau)?
- We know the income for the **population 15+**



Source: U.S. Census Bureau, Mean Personal Income in the United States [MAPAINUSA646N], retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/MAPAINUSA646N, October 24, 2022. Shaded areas indicate U.S. recessions.

Example 3: Census & GSS

- We know the income for the **2021 GSS sample of Veterans**

```
. mean conrinc if veteran==1

Mean estimation                              Number of obs = 229
```

	Mean	Std. err.	[95% conf.	interval]
conrinc	49562.49	2932.717	43783.8	55341.19

- What causes the difference between $57,143.00 (pop.15+, Census) and $49,562.49 (Veterans, GSS)?
- Real difference? Or difference due to random chance?

Example 3: Result

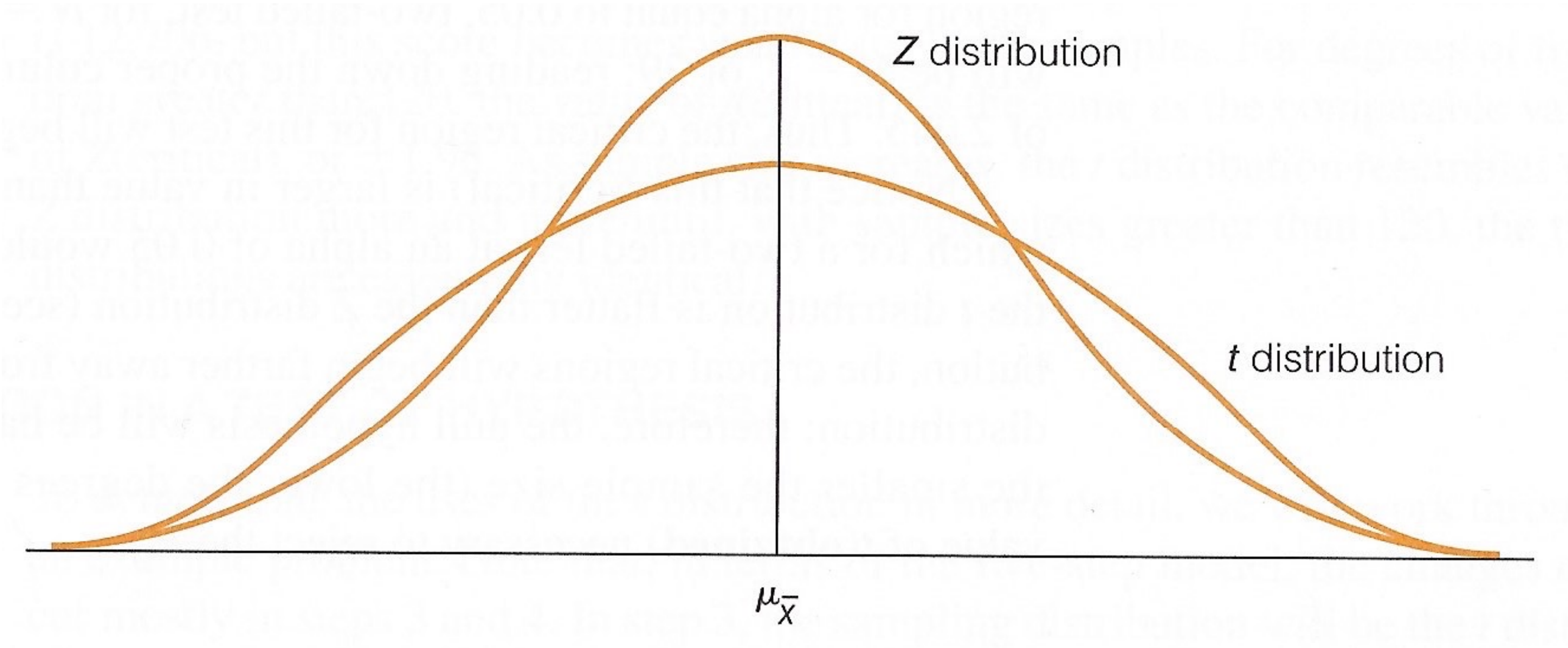
- Veteran population has mean income that is significantly lower than mean income of the population 15+
  - The difference between the parameter \$57,143.00 and the statistic \$49,562.49 was large and unlikely to have occurred by random chance (*p*-value<0.05)

```
. ztest conrinc=57143 if veteran==1

One-sample z test
------------------------------------------------------------------------------
Variable |     Obs        Mean    Std. err.   Std. dev.   [95% conf. interval]
---------+--------------------------------------------------------------------
 conrinc |     229    49562.49    .0660819           1    49562.36    49562.62
------------------------------------------------------------------------------
    mean = mean(conrinc)                                          z = -1.1e+05
H0: mean = 57143

   Ha: mean < 57143           Ha: mean != 57143            Ha: mean > 57143
 Pr(Z < z) = 0.0000       Pr(|Z| > |z|) = 0.0000        Pr(Z > z) = 1.0000
```

The Student's *t* distribution

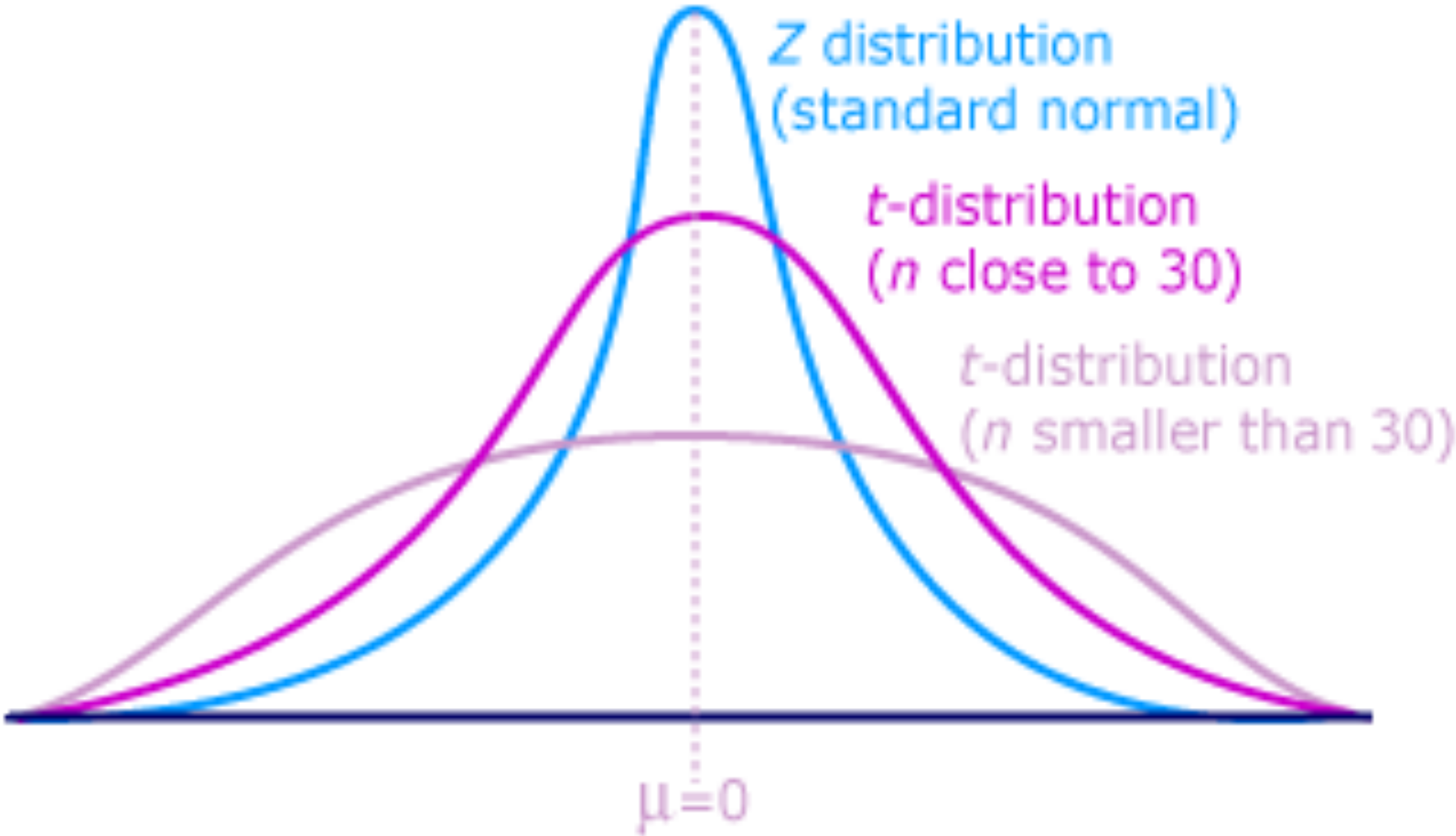
- How can we test a hypothesis when the population standard deviation (σ) is unknown, as is usually the case?
- For large samples ($n \geq 100$), we can use the sample standard deviation (s) as an estimator of the population standard deviation (σ)
  - Use standard normal distribution (Z)
- For small samples, s is too biased to estimate σ
  - Do not use standard normal distribution
  - Use Student's *t* distribution

t and *Z* distributions



Source: Healey 2015, p.203.

t and *Z* distributions



Source: https://joejeong33.wordpress.com/2013/06/03/t-distributionin-the-normal-distribution-there-are-enough/.

Choosing the distribution

- Choosing a sampling distribution when testing single-sample means for significance

If population standard deviation (σ) is	Sampling distribution is the
Known	*Z* distribution
Unknown and sample size (*n*) is large	*Z* distribution
Unknown and sample size (*n*) is small	*t* distribution

Source: Healey 2015, p.207.

Example 4: With *t*-test

- This is the same as example 3, but with *t*-test
  - GSS has a large sample. This is just an illustration

- Veteran population has mean income that is significantly lower than mean income of the population 15+ (p-value<0.05)

```
. ttest conrinc=57143 if veteran==1

One-sample t test
------------------------------------------------------------------------------
Variable |     Obs        Mean    Std. err.   Std. dev.   [95% conf. interval]
---------+--------------------------------------------------------------------
 conrinc |     229    49562.49    2932.717    44380.07     43783.8    55341.19
------------------------------------------------------------------------------
    mean = mean(conrinc)                                          t =  -2.5848
H0: mean = 57143                                 Degrees of freedom =      228

   Ha: mean < 57143           Ha: mean != 57143           Ha: mean > 57143
 Pr(T < t) = 0.0052       Pr(|T| > |t|) = 0.0104         Pr(T > t) = 0.9948
```

Five-step model for proportions

- When analyzing variables that are not measured at the interval-ratio level
  - A mean is inappropriate
  - We can test a hypothesis on a one sample proportion
- The five step model remains primarily the same, with the following changes
  - The assumptions are: random sampling, nominal level of measurement, and normal sampling distribution
  - The formula for *Z* is

$$Z = \frac{P_s - P_u}{\sqrt{P_u(1 - P_u)/n}}$$

Example 5: Proportions

- A random sample of 122 households in a low-income neighborhood revealed that 53 of the households were headed by women
  - $P_s = 53 / 122 = 0.43$
- In the city as a whole, the proportion of women-headed households (P_u) is 0.39
- Are households in lower-income neighborhoods significantly different from the city as a whole?
- Conduct a 90% hypothesis test ($\alpha = 0.10$)

Step 1: Assumptions,requirements

- Make assumptions
  - Random sampling
  - Hypothesis testing assumes samples were selected according to EPSEM
- Meet test requirements
  - The sample of 122 was randomly selected from all lower-income neighborhoods
  - Level of measurement is nominal
    - Women-headed households is measured as a proportion
  - Sampling distribution is normal in shape
    - This is a large sample ($n \geq 100$)

Step 2: Null hypothesis

- Null hypothesis, H_0: $P_u = 0.39$
  - The sample of 122 comes from a population where 39% of households are headed by women
  - The difference between 0.43 and 0.39 is trivial and caused by random chance
- Alternative hypothesis, H_1: $P_u \neq 0.39$
  - The sample of 122 comes from a population where the percent of women-headed households is not 39
  - The difference between 0.43 and 0.39 reflects an actual difference between lower-income neighborhoods and all neighborhoods

Step 3: Distribution, critical region

- Sampling distribution
  - Standard normal distribution (*Z*)
- Alpha (*α*) = 0.10 (two-tailed)
- Critical region begins at *Z(critical)* = ±1.65
  - This is the critical *Z* score associated with a two-tailed test and alpha equal to 0.10
  - If the obtained *Z* score falls in the critical region, we reject H_0

Step 4: Test statistic

- Proportion of households headed by women

City	Sample in a low-income neighborhood
P_u = 0.39	P_s = 0.43
	$n = 122$

- The formula for Z is

$$Z = \frac{P_s - P_u}{\sqrt{P_u(1 - P_u)/n}} = \frac{0.43 - 0.39}{\sqrt{0.39(1 - 0.39)/122}} = 0.91$$

Step 5: Decision, interpret

- *Z(obtained)* = 0.91
  - *Z(obtained)* did not fall in the critical region delimited by *Z(critical)* = ±1.65, so we ***do not reject*** the H_0
  - This means that if H_0 was true, a sample outcome of 0.43 would be likely
  - Therefore, the H_0 is not false and cannot be rejected
- The population of women-headed households in lower-income neighborhoods is not significantly different from the city as a whole
  - The difference between the parameter (P_u=0.39) and the statistic (P_s=0.43) was small and likely to have occurred by random chance (p>0.10)

Example 6: Sex, 2021

- Is the female proportion of the adult population (18+) in the U.S. higher than among the total population?
- We know the percentage of women for the **population**

PEOPLE	
Population	
Population Estimates, July 1 2021, (V2021)	**331,893,745**
Population estimates base, April 1, 2020, (V2021)	331,449,281
Population, percent change - April 1, 2020 (estimates base) to July 1, 2021, (V2021)	0.1%
Population, Census, April 1, 2020	331,449,281
Population, Census, April 1, 2010	308,745,538
Age and Sex	
Persons under 5 years, percent	5.7%
Persons under 18 years, percent	22.2%
Persons 65 years and over, percent	16.8%
Female persons, percent	50.5%

Source: U.S. Census Bureau (https://www.census.gov/quickfacts/fact/table/US/PST045221).

Example 6: Census & GSS

- The percentage of women in the **<u>2021 GSS sample 18+</u>**

```
. tab female
```

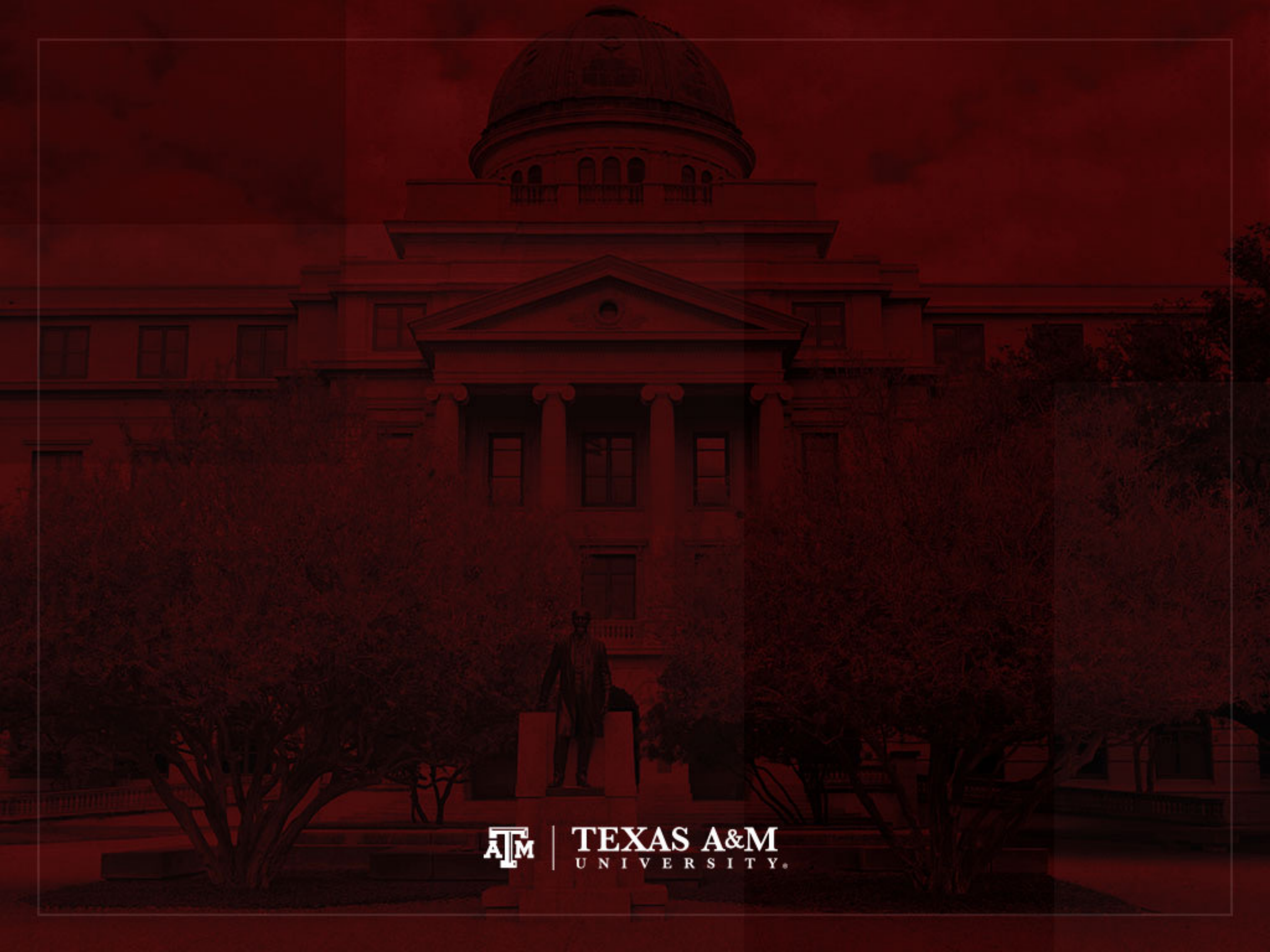
female	Freq.	Percent	Cum.
0	1,736	44.06	44.06
1	2,204	55.94	100.00
Total	3,940	100.00	

- What causes the difference between 50.5% (population, Census) and 55.94% (sample 18+, GSS)?
- Real difference? Or difference due to random chance?

Example 6: Result

- Population 18+ has a statistically significant higher proportion of women than overall population
  - The difference between the parameter 50.5% and the statistic 55.94% was large and unlikely to have occurred by random chance (*p*-value<0.05)

```
. prtest female=.505

One-sample test of proportion                   Number of obs      =      3940

------------------------------------------------------------------------------
    Variable |       Mean   Std. err.                     [95% conf. interval]
-------------+----------------------------------------------------------------
      female |   .5593909   .0079093                       .543889    .5748927
------------------------------------------------------------------------------
    p = proportion(female)                                        z =   6.8285
H0: p = 0.505

    Ha: p < 0.505                 Ha: p != 0.505                 Ha: p > 0.505
 Pr(Z < z) = 1.0000         Pr(|Z| > |z|) = 0.0000          Pr(Z > z) = 0.0000
```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