Lecture 9: Hypothesis testing: Two-sample case

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Source: Healey, Joseph F. 2015. "Statistics: A Tool for Social Research." Stamford: Cengage Learning. 10th edition. Chapter 9 (pp. 216–246).

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Outline

- Identify and cite examples of situations in which the two-sample test of hypothesis is appropriate
- Explain the logic of hypothesis testing, as applied to the two-sample case
- Explain what an independent random sample is
- Perform a test of hypothesis for two sample means or two sample proportions, following the five-step model and correctly interpret the results
- List and explain each of the factors (especially sample size) that affect the probability of rejecting the null hypothesis
- Explain the differences between statistical significance and importance

Basic logic

- We analyze a difference between two sample statistics
  - We compare means or proportions of two samples from specific sub-groups of the population

- This is the question under consideration
  - "Is the difference between the samples large enough to allow us to conclude (with a known probability of error) that the populations represented by the samples are different?"

Source: Healey 2015, p.217.

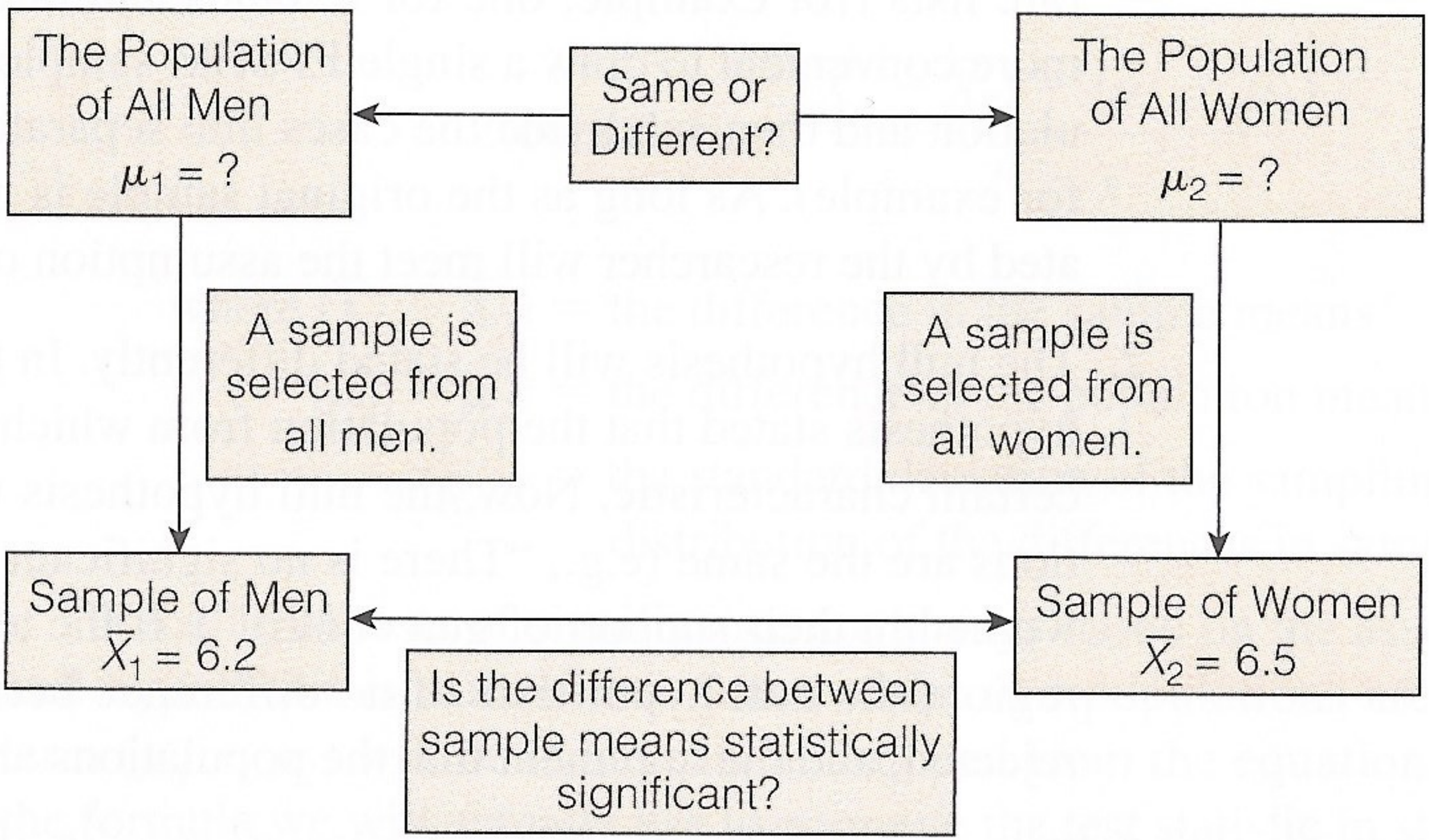
Null hypothesis

- The H_0 indicates that the populations are the same
  - Assuming that the H_0 is true, there is no difference between the parameters of the two populations
- On the other hand, we reject the H_0 and say there is a difference between the populations
  - If the difference between the sample statistics is large enough
  - Or if the size of the estimated difference is unlikely

H_0, *α*, *Z* score, *p*-value

- The H_0 is a statement of "no difference"
- The 0.05 level (*α*) will continue to be our indicator of a significant difference
- We change the sample statistics to a *Z* score
  - Place the *Z(obtained)* on the sampling distribution
- Estimate probability (*p*-value) above *Z(obtained)*
  - *p*-value is the probability of not rejecting the null hypothesis
  - Compare the *p*-value to the *α*
  - If $p<\alpha$, we reject H_0
  - If $p>\alpha$, we do not reject H_0

Test of hypothesis for two sample means



Source: Healey 2015, p.217.

The five-step model

1. Make assumptions and meet test requirements
2. Define the null hypothesis (H_0)
3. Select the sampling distribution and establish the critical region
4. Compute the test statistic
5. Make a decision and interpret the test results

Changes from one-sample case

- Step 1
  - In addition to samples selected according to EPSEM principles
  - Samples must be selected independently of each other: independent random sampling
- Step 2
  - Null hypothesis statement will state that the two populations are not different
- Step 3
  - Sampling distribution refers to difference between the sample statistics

Two-sample test of means (large samples)

- Do men and women significantly differ on their support of gun control?
- For men (sample 1)
  - Mean = 6.2
  - Standard deviation = 1.3
  - Sample size = 324
- For women (sample 2)
  - Mean = 6.5
  - Standard deviation = 1.4
  - Sample size = 317

Step 1: Assumptions,requirements

- Independent random sampling
  - The samples must be independent of each other
- Level of measurement is interval-ratio
  - Support of gun control is assessed with an interval-ratio level scale, so the mean is an appropriate statistic
- Sampling distribution is normal in shape
  - Total $n \geq 100$ ($n_1 + n_2 = 324 + 317 = 641$)
  - Thus, the Central Limit Theorem applies and we can assume a standard normal distribution (Z)

Step 2: Null hypothesis

- Null hypothesis, H_0: $\mu_1 = \mu_2$
  - The null hypothesis asserts there is no difference between the populations

- Alternative hypothesis, H_1: $\mu_1 \neq \mu_2$
  - The research hypothesis contradicts the H_0 and asserts there is a difference between the populations

Step 3: Distribution, critical region

- Sampling distribution
  - Standard normal distribution (*Z*)
- Significance level
  - Alpha (α) = 0.05 (two-tailed)
  - The decision to reject the null hypothesis has only a 0.05 probability of being incorrect
- *Z(critical)* = ±1.96
  - If the probability (*p*-value) is less than 0.05
  - *Z(obtained)* will be beyond *Z(critical)*

Step 4: Test statistic

- Sample outcomes for support of gun control

Sample 1 (men)	Sample 2 (women)
$\bar{X}_1 = 6.2$	$\bar{X}_2 = 6.5$
$s_1 = 1.3$	$s_2 = 1.4$
$n_1 = 324$	$n_2 = 317$

- Pooled estimate of the standard error

$$\sigma_{\bar{X}_1 - \bar{X}_2} = \sqrt{\frac{s_1^2}{n_1 - 1} + \frac{s_2^2}{n_2 - 1}} = \sqrt{\frac{(1.3)^2}{324 - 1} + \frac{(1.4)^2}{317 - 1}} = 0.107$$

- Obtained Z score

$$Z(obtained) = \frac{\bar{X}_1 - \bar{X}_2}{\sigma_{\bar{X}_1 - \bar{X}_2}} = \frac{6.2 - 6.5}{0.107} = -2.80$$

Step 5: Decision, interpret

- *Z(obtained)* = –2.80
  - This is beyond *Z(critical)* = ±1.96
  - The obtained *Z* score falls in the critical region, so we ***reject*** the H_0
  - Therefore, the H_0 is false and must be rejected

- The difference between men's and women's support of gun control is statistically significant
  - The difference between the sample means is so large that we can conclude (at $\alpha = 0.05$) that a difference exists between the populations represented by the samples

Two-sample test of means
(small samples)

- Do families that reside in the center-city have more children than families that reside in the suburbs?
- For suburbs (sample 1)
  - Mean = 2.37
  - Standard deviation = 0.63
  - Sample size = 42
- For center-city (sample 2)
  - Mean = 2.78
  - Standard deviation = 0.95
  - Sample size = 37

Step 1: Assumptions,requirements

- Independent random sampling
  - The samples must be independent of each other
- Level of measurement is interval-ratio
  - Number of children can be treated as interval-ratio
- Population variances are equal
  - As long as the two samples are approximately the same size, we can make this assumption
- Sampling distribution is normal in shape
  - Because we have two small samples (n < 100), we have to add the previous assumption in order to meet this assumption

Step 2: Null hypothesis

- Null hypothesis, H_0: $\mu_1 = \mu_2$
  - The null hypothesis asserts there is no difference between the populations

- Alternative hypothesis, H_1: $\mu_1 < \mu_2$
  - The research hypothesis contradicts the H_0 and asserts there is a difference between the populations

Step 3: Distribution, critical region

- Sampling distribution
  - Student's *t* distribution
- Significance level
  - Alpha (α) = 0.05 (one-tailed)
- Degrees of freedom
  - $n_1 + n_2 - 2 = 42 + 37 - 2 = 77$
- Critical *t*
  - t(critical) = –1.671

Step 4: Test statistic

- Sample outcomes for number of children

Sample 1 (suburban)	Sample 2 (center-city)
$\bar{X}_1 = 2.37$	$\bar{X}_2 = 2.78$
$s_1 = 0.63$	$s_2 = 0.95$
$n_1 = 42$	$n_2 = 37$

- Pooled estimate of the standard error

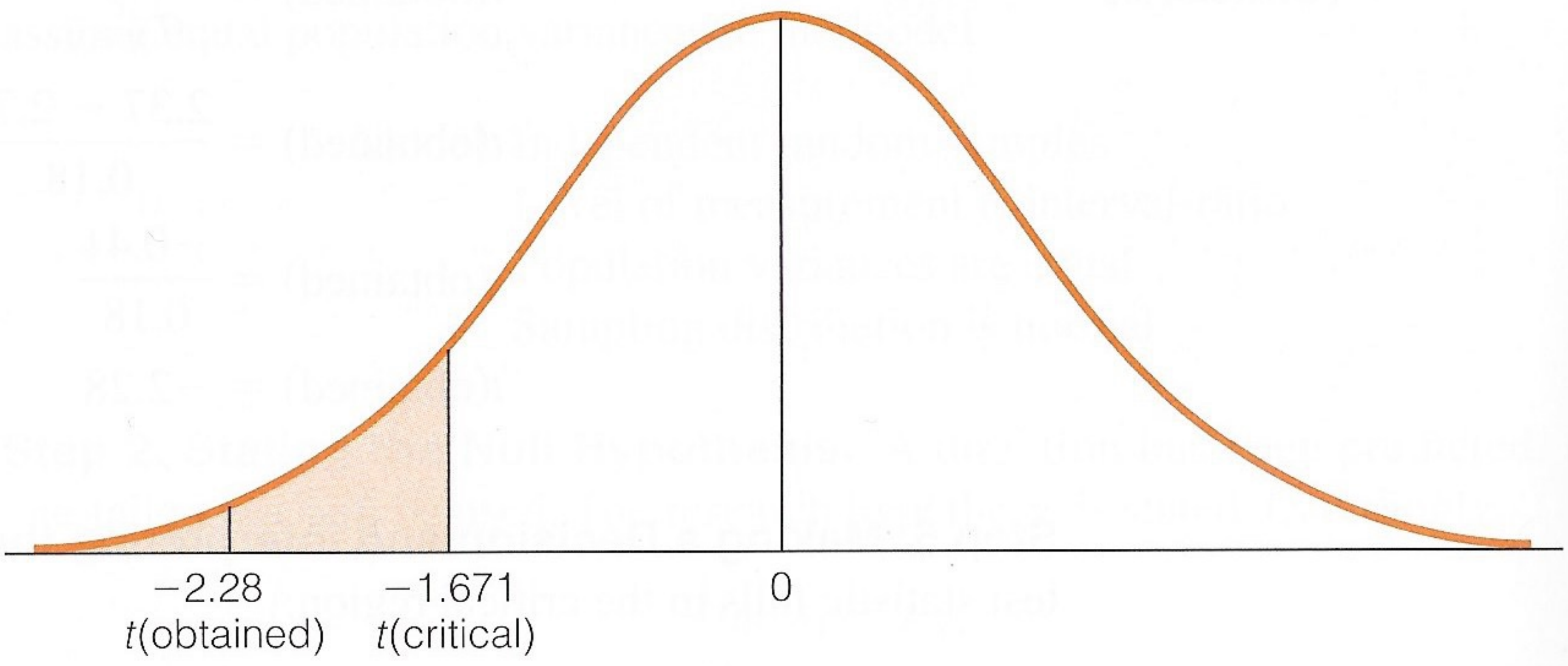
$$\sigma_{\bar{X}_1-\bar{X}_2} = \sqrt{\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}}\sqrt{\frac{n_1 + n_2}{n_1 n_2}} = \sqrt{\frac{(42)(0.63)^2 + (37)(0.95)^2}{42 + 37 - 2}}\sqrt{\frac{42 + 37}{(42)(37)}} = 0.18$$

- Obtained *t*

$$t(obtained) = \frac{\bar{X}_1 - \bar{X}_2}{\sigma_{\bar{X}_1-\bar{X}_2}} = \frac{2.37 - 2.78}{0.18} = -2.28$$

t(obtained) & *t*(critical)

- Sampling distribution with critical region and test statistic displayed



Source: Healey 2015, p.226.

Step 5: Decision, interpret

- t(obtained) = –2.28
  - This is beyond t(critical) = –1.671
  - The obtained test statistic falls in the critical region, so we ***reject*** the H_0

- The difference between the number of children in center-city families and the suburban families is statistically significant
  - The difference between the sample means is so large that we can conclude (at α = 0.05) that a difference exists between the populations represented by the samples

Example from GSS: *t*-test

- We know the average income by sex from the 2016 GSS

```
. table sex, c(mean conrinc)
```

responden ts sex	mean(conrinc)
male	**41583.52814**
female	**28353.34628**

- What causes the difference between male income of \$41,583.53 and female income of \$28,353.35?
- Real difference? Or difference due to random chance?

Example from GSS: Result

- Men have an average income that is significantly higher than the female average income
  - The difference between male income ($41,583.53) and female income ($28,353.35) was large and unlikely to have occurred by random chance (*p*<0.05) in 2016

```
. ttest conrinc, by(sex)

Two-sample t test with equal variances
------------------------------------------------------------------------------
   Group |     Obs        Mean    Std. Err.   Std. Dev.   [95% Conf. Interval]
---------+--------------------------------------------------------------------
    male |     798    41583.53    1433.963    40507.87    38768.74    44398.32
  female |     834    28353.35    1049.496    30308.45    26293.38    30413.31
---------+--------------------------------------------------------------------
combined |   1,632    34822.52    897.5571    36259.53    33062.03       36583
---------+--------------------------------------------------------------------
    diff |            13230.18    1765.955                9766.402    16693.96
------------------------------------------------------------------------------
    diff = mean(male) - mean(female)                              t =   7.4918
Ho: diff = 0                                     degrees of freedom =     1630

    Ha: diff < 0                 Ha: diff != 0                 Ha: diff > 0
 Pr(T < t) = 1.0000         Pr(|T| > |t|) = 0.0000          Pr(T > t) = 0.0000
```

Edited table

Table 1. Two-sample *t*-test of individual average income of the U.S. adult population by sex, 2004, 2010, and 2016

Sex	2004	2010	2016
Male	45,741.48	37,864.34	41,583.53
	(1,343.92)	(1,359.39)	(1,433.96)
Female	29,264.54	26,141.60	28,353.35
	(972.15)	(972.97)	(1,049.50)
Difference	16,476.94***	11,722.74***	13,230.18***
	(1,665.71)	(1,643.94)	(1,765.96)
Sample size	1,688	1,202	1,632

Note: Standard errors are reported in parentheses. *Significant at p<0.10; **Significant at p<0.05; ***Significant at p<0.01.
Source: 2004, 2010, 2016 General Social Surveys.

Two-sample test of proportions (large samples)

- Do Black and White senior citizens differ in their number of memberships in clubs and organizations?
  - Using the proportion of each group classified as having a “high” level of membership
- For Black senior citizens (sample 1)
  - Proportion = 0.34
  - Sample size = 83
- For White senior citizens (sample 2)
  - Proportion = 0.25
  - Sample size = 103

Step 1: Assumptions,requirements

- Independent random sampling
  - The samples must be independent of each other
- Level of measurement is nominal
  - We have measured the proportion of each group classified as having a “high” level of membership
- Population variances are equal
  - As long as the two samples are approximately the same size, we can make this assumption
- Sampling distribution is normal in shape
  - Total $n \geq 100$ ($n_1 + n_2 = 83 + 103 = 186$)
  - Thus, the Central Limit Theorem applies and we can assume a standard normal distribution

Step 2: Null hypothesis

- Null hypothesis, H_0: $P_{u1} = P_{u2}$
  - The null hypothesis asserts there is no difference between the populations

- Alternative hypothesis, H_1: $P_{u1} \neq P_{u2}$
  - The research hypothesis contradicts the H_0 and asserts there is a difference between the populations

Step 3: Distribution, critical region

- Sampling distribution
  - Standard normal distribution (*Z*)

- Significance level
  - Alpha (α) = 0.05 (two-tailed)
  - The decision to reject the null hypothesis has only a 0.05 probability of being incorrect

- *Z(critical)* = ±1.96
  - If the probability (*p*-value) is less than 0.05
  - *Z(obtained)* will be beyond *Z(critical)*

Step 4: Test statistic

- Sample outcomes for club memberships

Sample 1 (Black senior citizens)	Sample 2 (White senior citizens)
$P_{s1} = 0.34$	$P_{s2} = 0.25$
$n_1 = 83$	$n_2 = 103$

- Population proportion

$$P_u = \frac{n_1 P_{s1} + n_2 P_{s2}}{n_1 + n_2} = \frac{(83)(0.34) + (103)(0.25)}{83 + 103} = 0.29$$

- Pooled estimate of the standard error

$$\sigma_{p_1 - p_2} = \sqrt{P_u(1 - P_u)}\sqrt{\frac{n_1 + n_2}{n_1 n_2}} = \sqrt{(0.29)(0.71)}\sqrt{\frac{83 + 103}{(83)(103)}} = 0.07$$

- Obtained Z score

$$Z(obtained) = \frac{P_{s1} - P_{s2}}{\sigma_{p_1 - p_2}} = \frac{0.34 - 0.25}{0.07} = 1.29$$

Step 5: Decision, interpret

- *Z(obtained)* = 1.29
  - This is below the *Z(critical)* = 1.96
  - The obtained test statistic does not fall in the critical region, so we ***do not reject*** the H_0

- The difference between the memberships of Black and White senior citizens is not significant
  - The difference between the sample means is small enough that we can conclude (at $\alpha = 0.05$) that no difference exists between the populations represented by the samples

Example from GSS: proportion

- We know the proportion of pro-immigrants by political party from the 2016 GSS

```
. table democrat, c(mean proimmig)
```

Political party	mean(proimmig)
Republicans	.117096
Democrats	.4559471

- What causes the difference between the percentage of Republicans who a pro-immigration (11.7%) and the percentage of Democrats who are pro-immigration (45.6%)?
  - Real difference? Or difference due to random chance?

Example from GSS: Result

- Republicans are less pro-immigration than Democrats
  - The difference between the percentage of Republicans who are pro-immigration (11.7%) and the percentage of Democrats who are pro-immigration (45.6%) was large and unlikely to have occurred by random chance (*p*<0.05) in 2016

```
. prtest proimmig, by(democrat)

Two-sample test of proportions            Republicans: Number of obs =      427
                                            Democrats: Number of obs =      454

------------------------------------------------------------------------------
    Variable |       Mean   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
 Republicans |    .117096   .0155602                      .0865987    .1475934
   Democrats |   .4559471   .0233749                      .4101332    .5017611
-------------+----------------------------------------------------------------
        diff |  -.3388511   .0280803                     -.3938875   -.2838147
             |  under Ho:   .0306428  -11.06   0.000
------------------------------------------------------------------------------
        diff = prop(Republicans) - prop(Democrats)                z = -11.0581
    Ho: diff = 0

    Ha: diff < 0                 Ha: diff != 0                 Ha: diff > 0
 Pr(Z < z) = 0.0000         Pr(|Z| > |z|) = 0.0000          Pr(Z > z) = 1.0000
```

Edited table

Table 2. Test of proportions of pro-immigrants among the U.S. adult population by political party, 2004, 2010, and 2016

Political Party	2004	2010	2016
Republican	0.0911	0.1429	0.1171
	(0.0124)	(0.0193)	(0.0156)
Democratic	0.2164	0.2761	0.4559
	(0.0178)	(0.0223)	(0.0234)
Difference	–0.1253***	–0.1333***	–0.3389***
	(0.0217)	(0.0295)	(0.0281)
Sample size	1,074	731	881

Note: Standard errors are reported in parentheses. *Significant at p<0.10; **Significant at p<0.05; ***Significant at p<0.01.
Source: 2004, 2010, 2016 General Social Surveys.

Statistical significance vs. importance (magnitude)

- As long as we work with random samples, we must conduct a test of significance
- Statistical significance is not the same thing as importance
  - Importance is also known as magnitude of the effect
- Differences that are otherwise trivial or uninteresting may be significant

Influence of sample size

- When working with large samples, even small differences may be statistically significant
- The larger the sample size (*n*)
  - The greater the value of the test statistic
  - The more likely it will fall in the critical region and be declared statistically significant
- In general, when working with random samples, statistical significance is a necessary but not a sufficient condition for importance

Sample size & test statistic

Test Statistics for Single-Sample Means Computed from Samples of Various Sizes ($\bar{X} = 80$, $\mu = 79$, $s = 5$ throughout)

Sample Size (N)	Computing the Test Statistic	Test Statistic, Z(Obtained)
50	$Z(\text{obtained}) = \frac{\bar{X} - \mu}{\sigma/\sqrt{N-1}} = \frac{80-79}{5/\sqrt{49}} = \frac{1}{0.71} =$	1.41
100	$Z(\text{obtained}) = \frac{\bar{X} - \mu}{\sigma/\sqrt{N-1}} = \frac{80-79}{5/\sqrt{99}} = \frac{1}{0.50} =$	2.00
500	$Z(\text{obtained}) = \frac{\bar{X} - \mu}{\sigma/\sqrt{N-1}} = \frac{80-79}{5/\sqrt{499}} = \frac{1}{0.22} =$	4.55
1000	$Z(\text{obtained}) = \frac{\bar{X} - \mu}{\sigma/\sqrt{N-1}} = \frac{80-79}{5/\sqrt{999}} = \frac{1}{0.16} =$	6.25
10,000	$Z(\text{obtained}) = \frac{\bar{X} - \mu}{\sigma/\sqrt{N-1}} = \frac{80-79}{5/\sqrt{9999}} = \frac{1}{0.05} =$	20.00

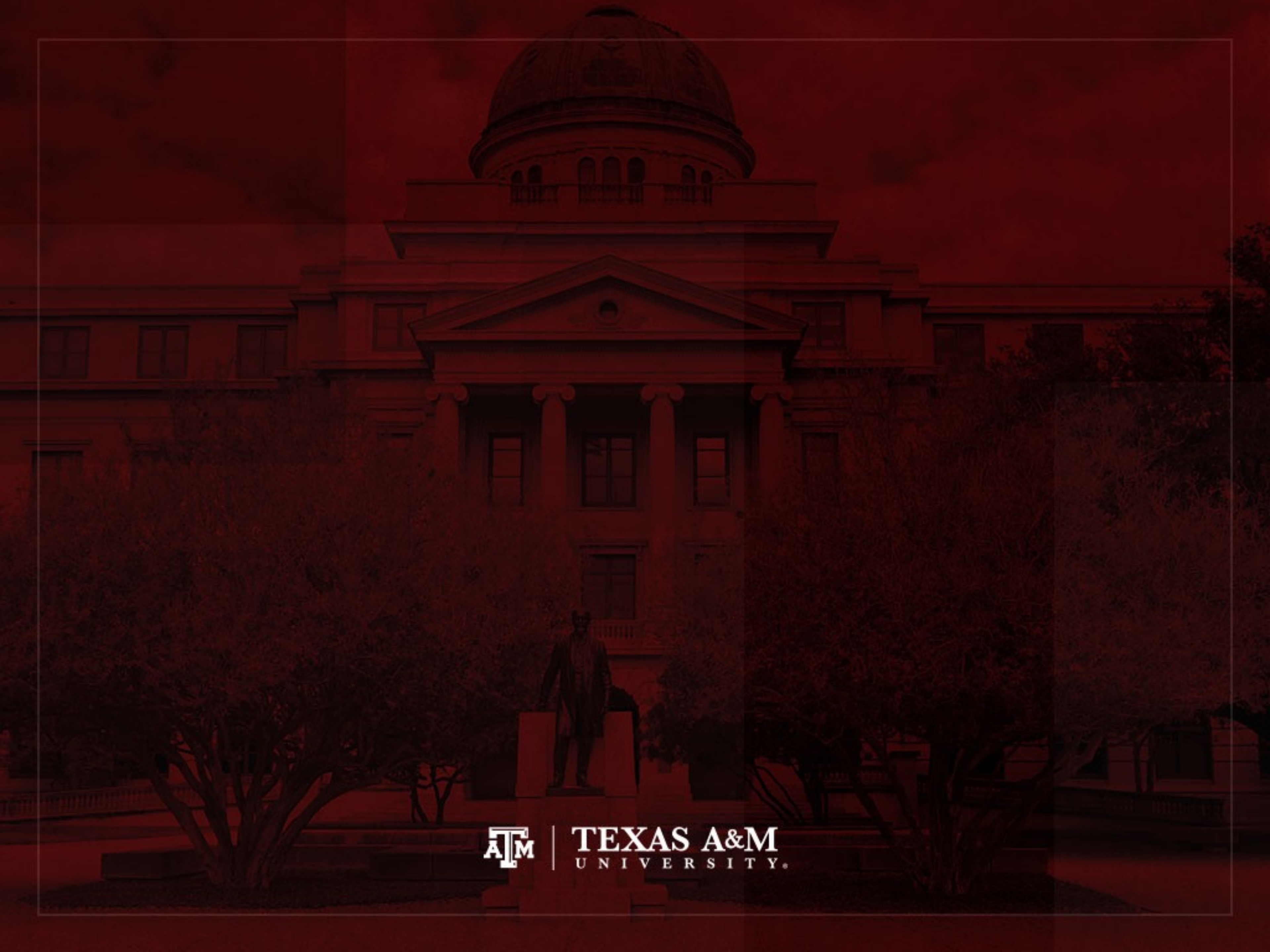
Source: Healey 2015, p.234.

Outcomes of hypothesis testing

- Result of a specific analysis could be
  - Statistically significant and
    - Important (large magnitude)
  - Statistically significant, but
    - Unimportant (small magnitude)
  - Not statistically significant, but
    - Important (large magnitude)
  - Not statistically significant and
    - Unimportant (small magnitude)

Factors influencing the decision

1. The size of the observed difference
   - For larger differences, we are more likely to reject H_0
2. The value of alpha
   - Usually the decision to reject the null hypothesis has only a 0.05 probability of being incorrect
   - The higher the alpha
     - The more likely we are to reject the H_0
     - But we would have a higher chance of being incorrect
3. The use of one- vs. two-tailed tests
   - We are more likely to reject H_0 with a one-tailed test
4. The size of the sample (*n*)
   - For larger samples, we are more likely to reject H_0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