Lecture 1b: Exponential growth

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Exponential growth

- Balancing equation
- Growth rate R
- Exponential function and curve
- Doubling times

Balancing equation

- Balancing equation for the world, 2010–2011

$$K(2011) = K(2010) + B(2010) – D(2010)$$

  - *K*(2010): world population at start of 2010
  - *B*(2010): births during 2010
  - *D*(2010): deaths during 2010
  - *K*(2011): population at start of 2011

World population 2010 to 2011

Population 1 January 2010	**6,851 million**
+ Births 2010	+140 million
+ Deaths 2010	–57 million
= Population 1 January 2011	6,934 million

Source: 2010 Population Data Sheet of the Population Reference Bureau (PRB). Wachter 2014, p. 6.

General form of balancing equation

- For closed population

$$K(t + n) = K(t) + B(t) - D(t)$$

  - n: length of a period, e.g. 1 year or 10 years
  - $B(t)$, $D(t)$: births, deaths during period from t to $t+n$

- Equation for national or regional populations are more complicated due to migration
  - Closed population examples are used to understand concepts

Pattern when combining equations

- Decompose next year's "stock" into this year's "stock" plus "flow"

$$K(1) = K(0) + [B(0) - D(0)]$$

- t=0 for present year, n=1 year long

Separate elements

- Multiply and divide by starting population *K*(0)

$$K(1) = K(0)\left(1 + \frac{B(0)}{K(0)} - \frac{D(0)}{K(0)}\right)$$

- Following year

$$K(2) = K(1)\left(1 + \frac{B(1)}{K(1)} - \frac{D(1)}{K(1)}\right)$$

- Substituting for *K*(1)

$$K(2) = \left(1 + \frac{B(1)}{K(1)} - \frac{D(1)}{K(1)}\right)\left(1 + \frac{B(0)}{K(0)} - \frac{D(0)}{K(0)}\right)K(0)$$

From starting to later population

- We go from a starting population to a later population by using multiplication
  - *B*/*K* and *D*/*K* are less dependent on *K* than *B* and *D*
  - Population growth appears as a multiplicative process
- Multiplicative growth ≈ geometric growth
  - Geometric growth applies for growth through whole time intervals
- Geometric growth = exponential growth
  - When fractions of intervals are involved, we use exponential function

Simple case

$$K(1) = K(0)\left(1 + \frac{B(0)}{K(0)} - \frac{D(0)}{K(0)}\right)$$

- When *B*/*K* and *D*/*K* are not changing or not changing much

$$A = 1 + \frac{B}{K} - \frac{D}{K}$$

- Equations take the form

$$K(1) = A\,K(0)$$

$$K(2) = A^2\,K(0)$$

...

$$K(T) = A^T\,K(0)$$

Example

- In 2000, there were 6.048 billion people, with births exceeding deaths by about 75 million/year

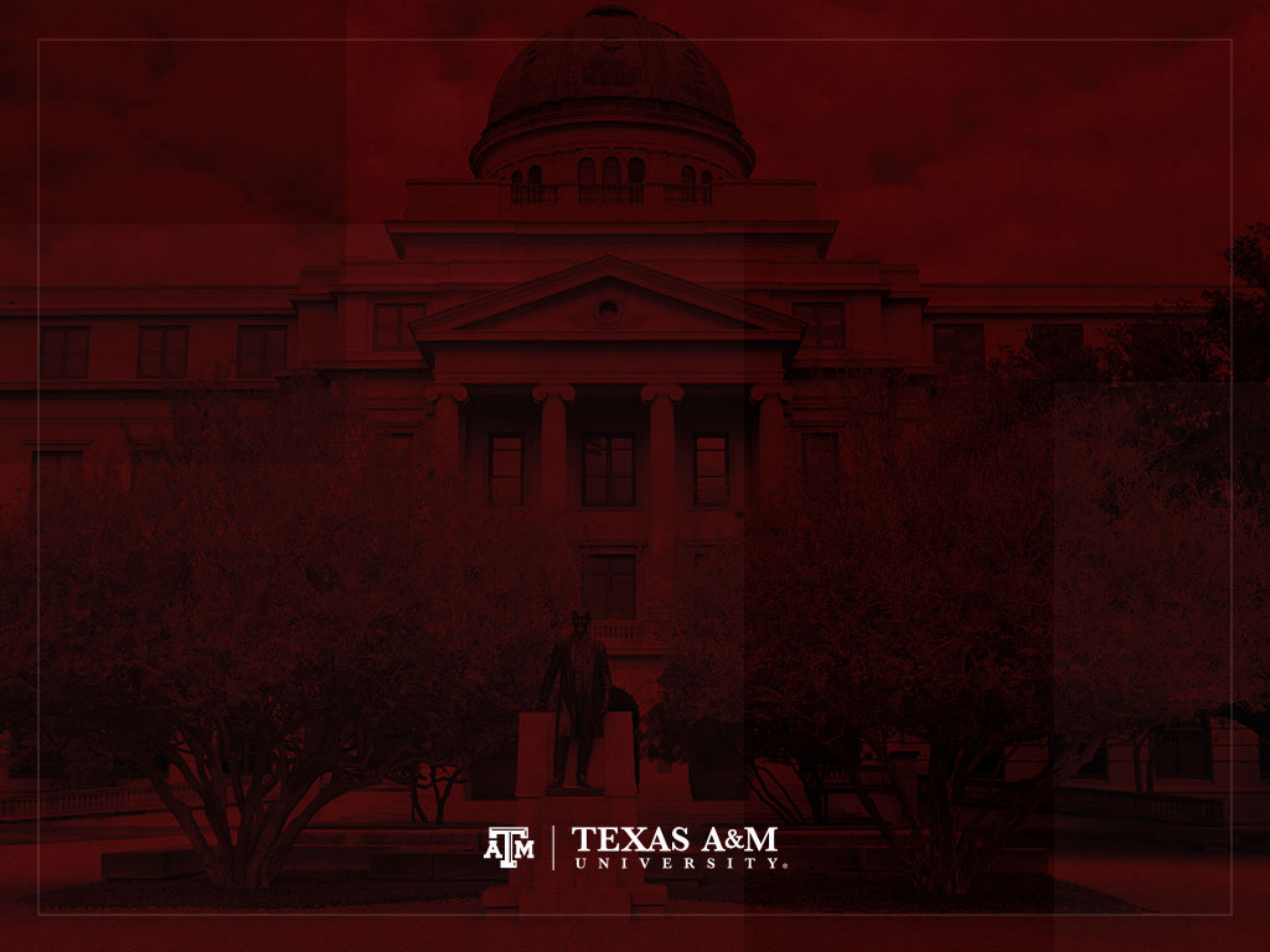
$$A = 1 + \frac{B}{K} - \frac{D}{K} = 1 + \frac{B - D}{K} = 1 + \frac{75}{6{,}048} = 1.0124$$

$K(0) = 1.0124^{0} * 6.048 = 6.048$

$K(1) = 1.0124^{1} * 6.048 = 6.123$

$K(10) = 1.0124^{10} * 6.048 = 6.841$

$K(12) = 1.0124^{12} * 6.048 = 7.012$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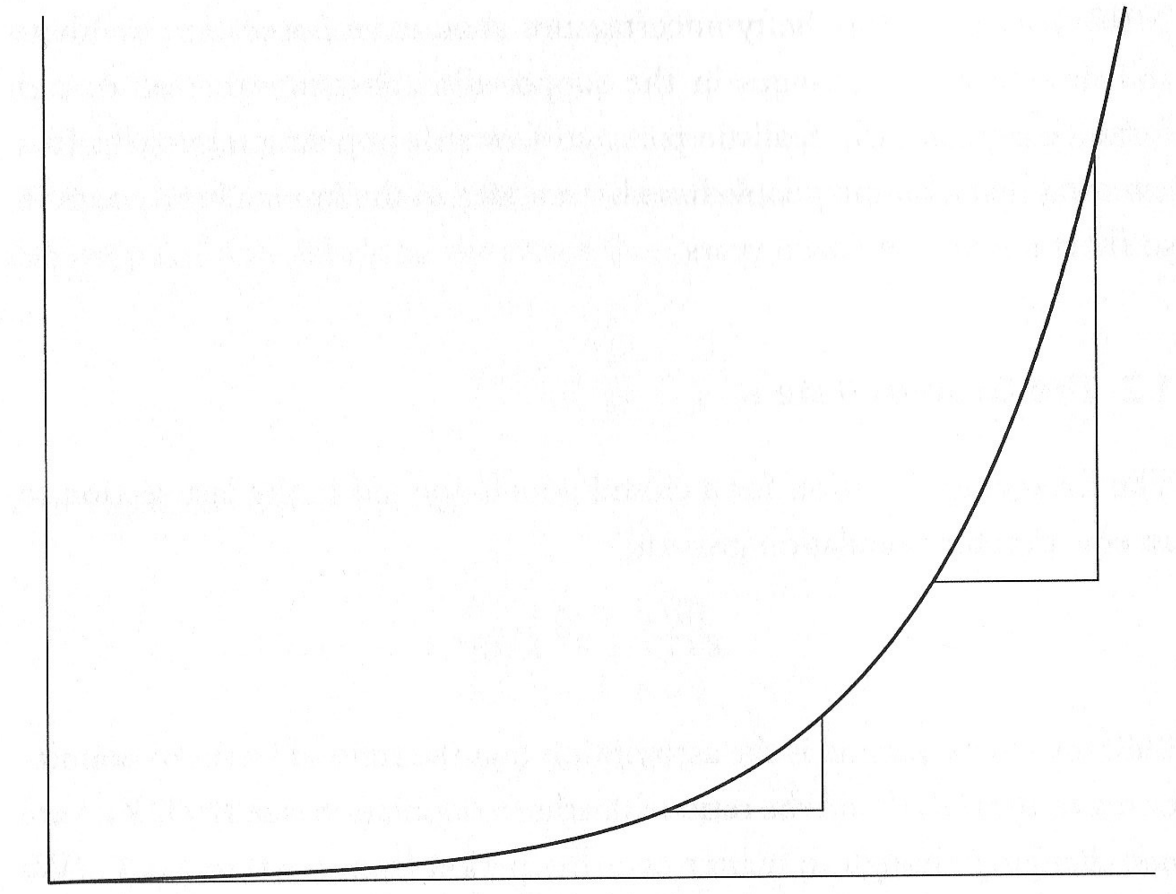
Growth rate R

- Balancing equation for closed population led to equation for population growth

$$K(T) = A^T K(0)$$

- $B(t)/K(t)$ and $D(t)/K(t)$ are not changing much
- When births exceed deaths, A is bigger than 1 and population increases
- Keeping same value of A through time, we get...

K(t) with ever-changing slope



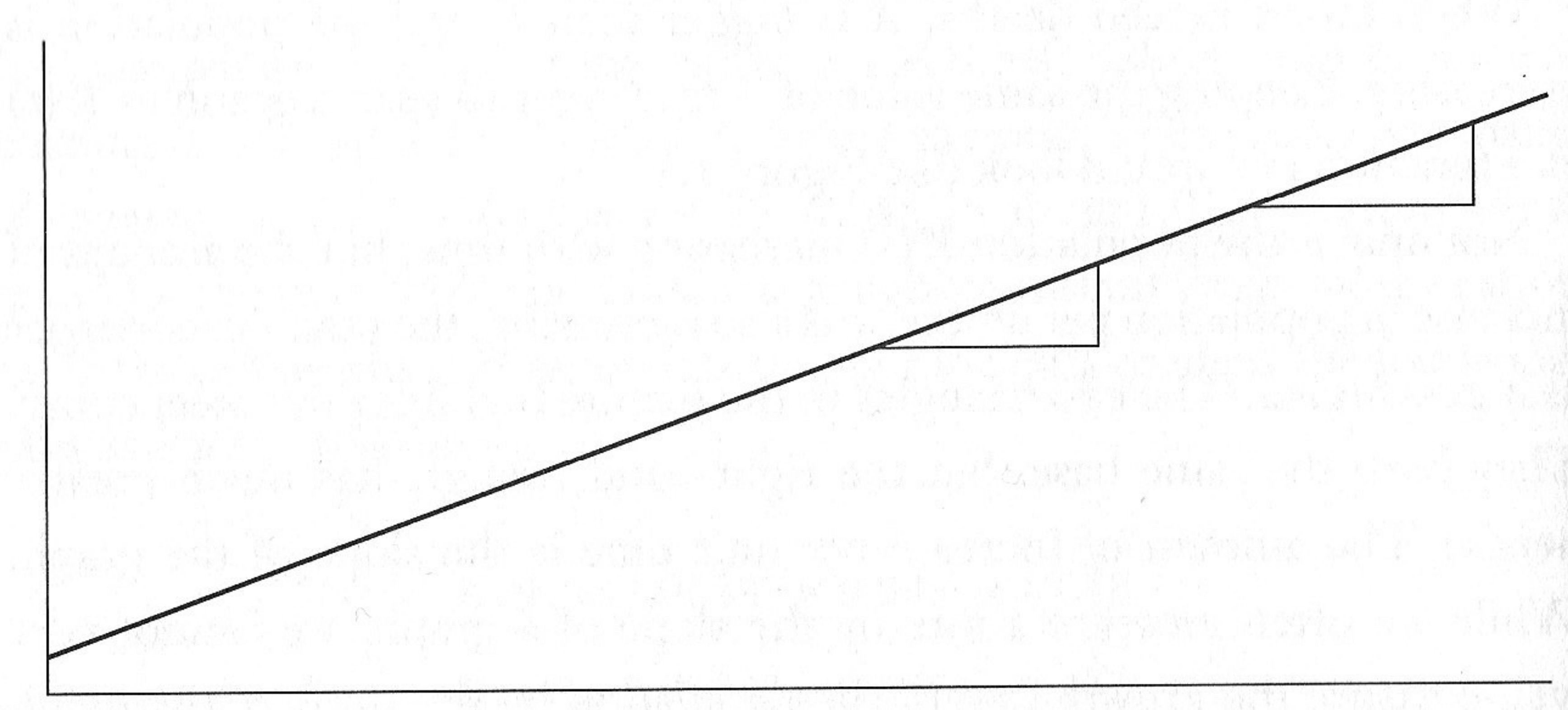
Source: Wachter 2014, p. 10.

Constant slope

- Previous graph, we cannot measure growth rate by graph slope, because it varies
  - Slope changes even when B/K and D/K are fixed

- We need a measure of growth that stays fixed when B/K and D/K are fixed
  - Take logarithms of $K(t)$
  - Usual way of converting multiplication into addition
  - $\log K(t)$ versus t has constant slope...

Log *K(t)* with constant slope



Source: Wachter 2014, p. 10.

Linear equation

- Taking logarithms converts the equation

$$K(t) = A^{t}\, K(0)$$

- Into the equation

$$\log(K(t)) = \log(K(0)) + \log(A)t$$

- General form of linear equation

$$\log(K(T)) = Y = a + bX$$

- Slope b is log(A), which is called slope R
  - Measure of population growth

Natural logarithms

- We use natural logarithms, which have base e=2.71828
  - "e" is the choice for A that makes the slope of the graph of K(t) equal 1 when t=0 and K(0)=1

- Population growth rate R
  - Slope of the graph of the logarithm of population size over time
  - Proportional rate of change in population size

Population growth rate (R)

- Growth is unchanging when the ratios of births and deaths to population size are unchanging
- The slope equals the ratio of change in vertical axis (rise) to horizontal axis (run)

$$R = \frac{\log\big(K(T)\big) - \log\left(K(0)\right)}{T - 0}$$

- It can also be written as

$$R = \frac{1}{T} log\left(\frac{K(T)}{K(0)}\right)$$

Average growth rate

- As slope of logarithm of population size

$$R = \frac{1}{T} log\left(1 + \frac{K(T) - K(0)}{K(0)}\right)$$

- As proportional rate of change in population size

$$R \approx \frac{K(T) - K(0)}{T} \frac{1}{K(0)}$$

  – When *T* (interval in years) is close to zero
  – First factor is ratio of vertical to horizontal axis
  – Divide it by *K*(0) to get slope as proportion of size

Summarizing growth rate

$$R = \frac{1}{n} log\left(\frac{K(t+n)}{K(t)}\right)$$

- *n*: length of a period, e.g. 1 year or 10 years
- *K*(*t*): population at the beginning of the interval
- *K*(*t*+*n*): population at the end of the interval

Summarizing growth rate

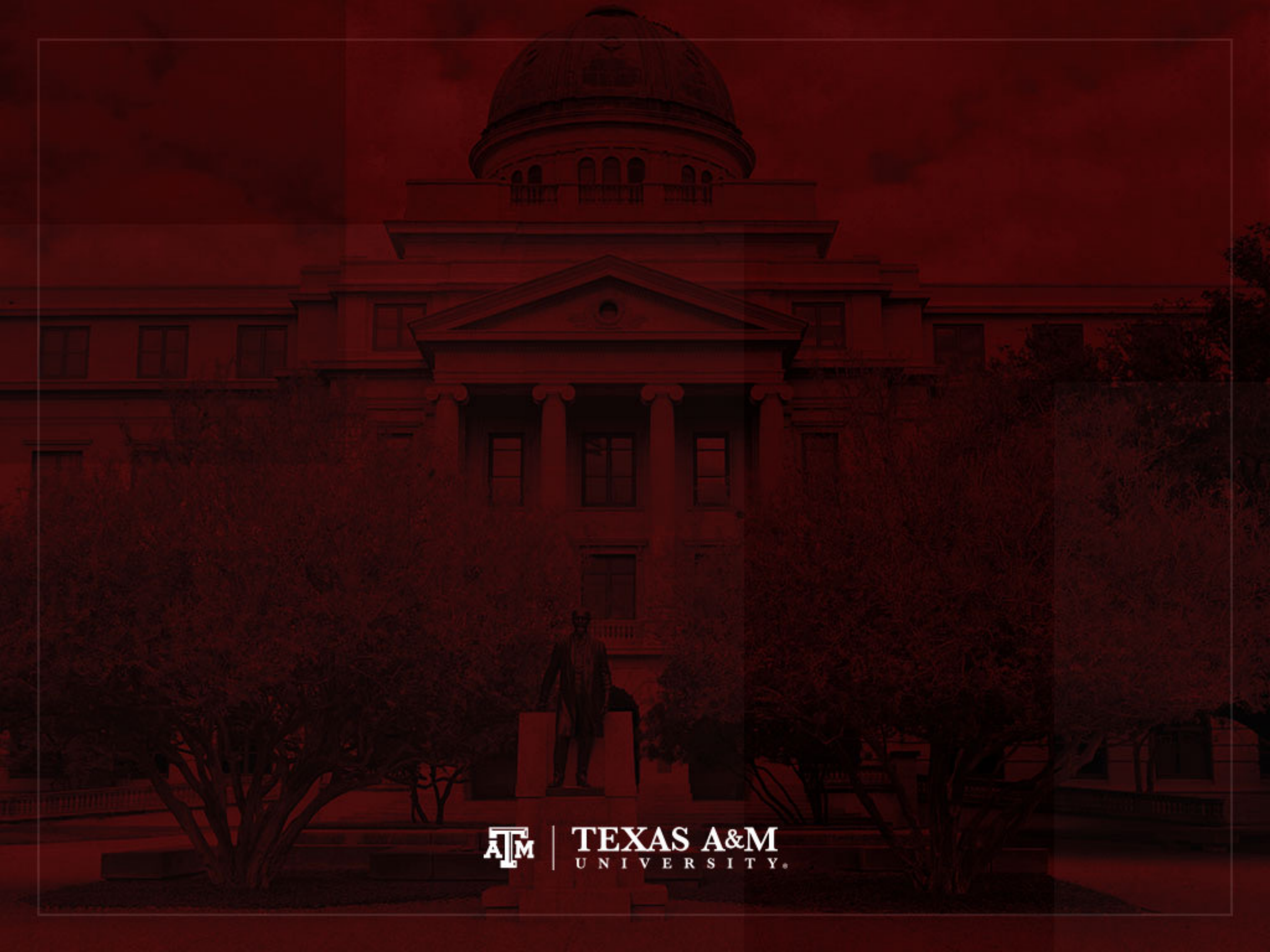
$$R = \frac{1}{t} ln\left(\frac{K(t)}{K(0)}\right)$$

- *n*: length of a period, e.g. 1 year or 10 years
- *K*(0): population at the beginning of the interval
- *K*(*t*): population at the end of the interval

Example of growth rate R

Population 1 January 2010	**6,851 million**
+ Births 2010	+140 million
+ Deaths 2010	–57 million
= Population 1 January 2011	6,934 million

- R = log[$K(t+n)/K(t)$] / n = log(6,934/6,851) / 1 = 0.012042
- R = log(1+(B–D)/K) = log(1+(140–57)/6,851) = 0.012042
- World population grew at a rate of about 12 per thousand per year between 2010 and 2011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Exponential function

- Population over time when ratios of births and deaths to population remain constant

$$K(t) = A^t K(0) = e^{Rt} K(0) = \exp(Rt)K(0)$$

- Exponential function is the inverse function for natural logarithms

$$e^{\log(x)} = \exp(\log(x)) = x$$

$$\log(e^y) = \log(\exp(y)) = y$$

Exponential curve

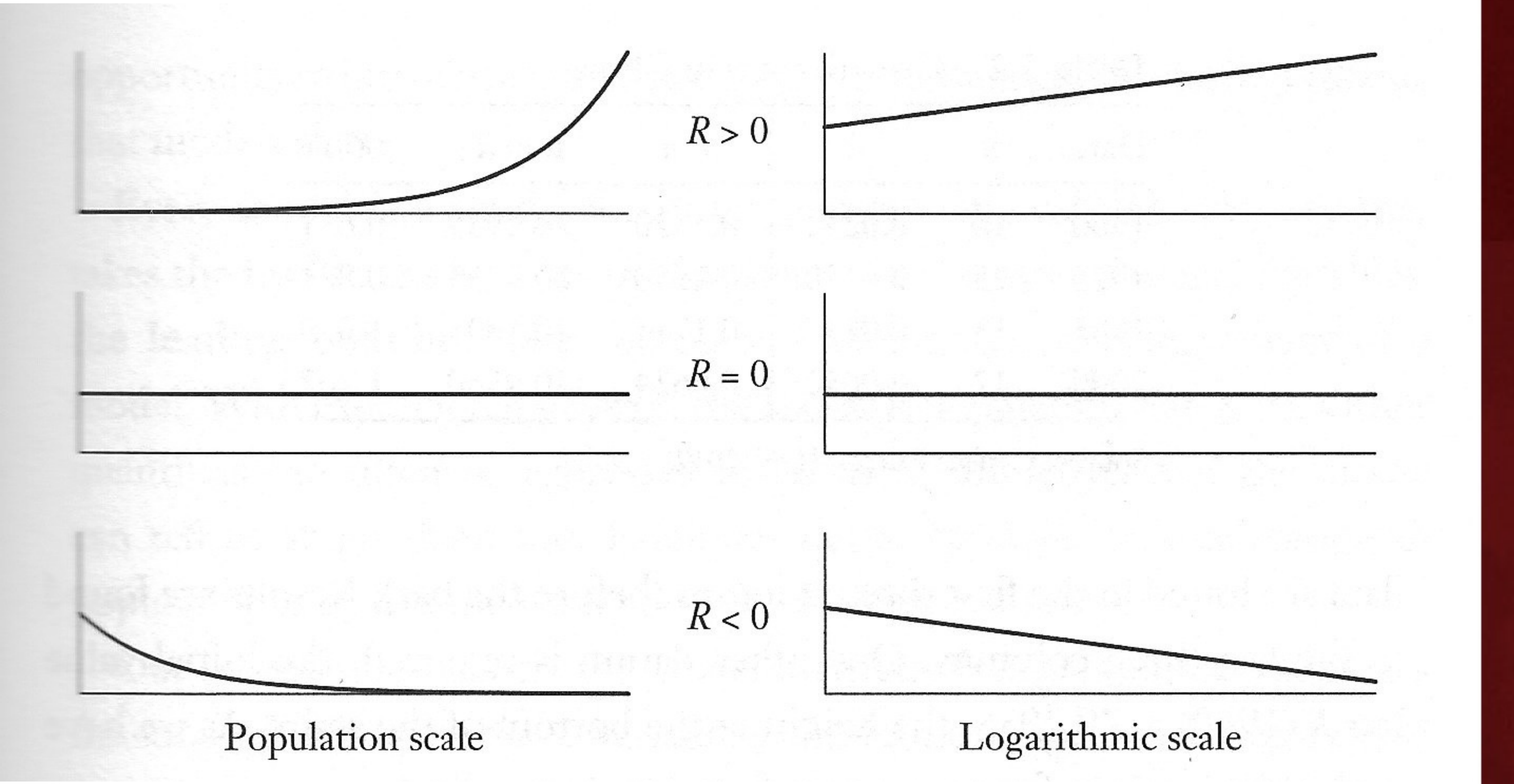
- We know that log(A) is R

$$A = e^{\log(A)} = e^{R}$$

$$A^{t} = (e^{R})^{t} = e^{Rt} = \exp(Rt)$$

- Exponential curve
  - It is the graph of $\exp(Rt)$ as a function of t
  - Continuous-time version of the curve for geometric growth

Trajectories of exponential growth



Source: Wachter 2014, p. 15.

Applying exponential function

- On the logarithm scale, our formula for growth at each step involves simple addition

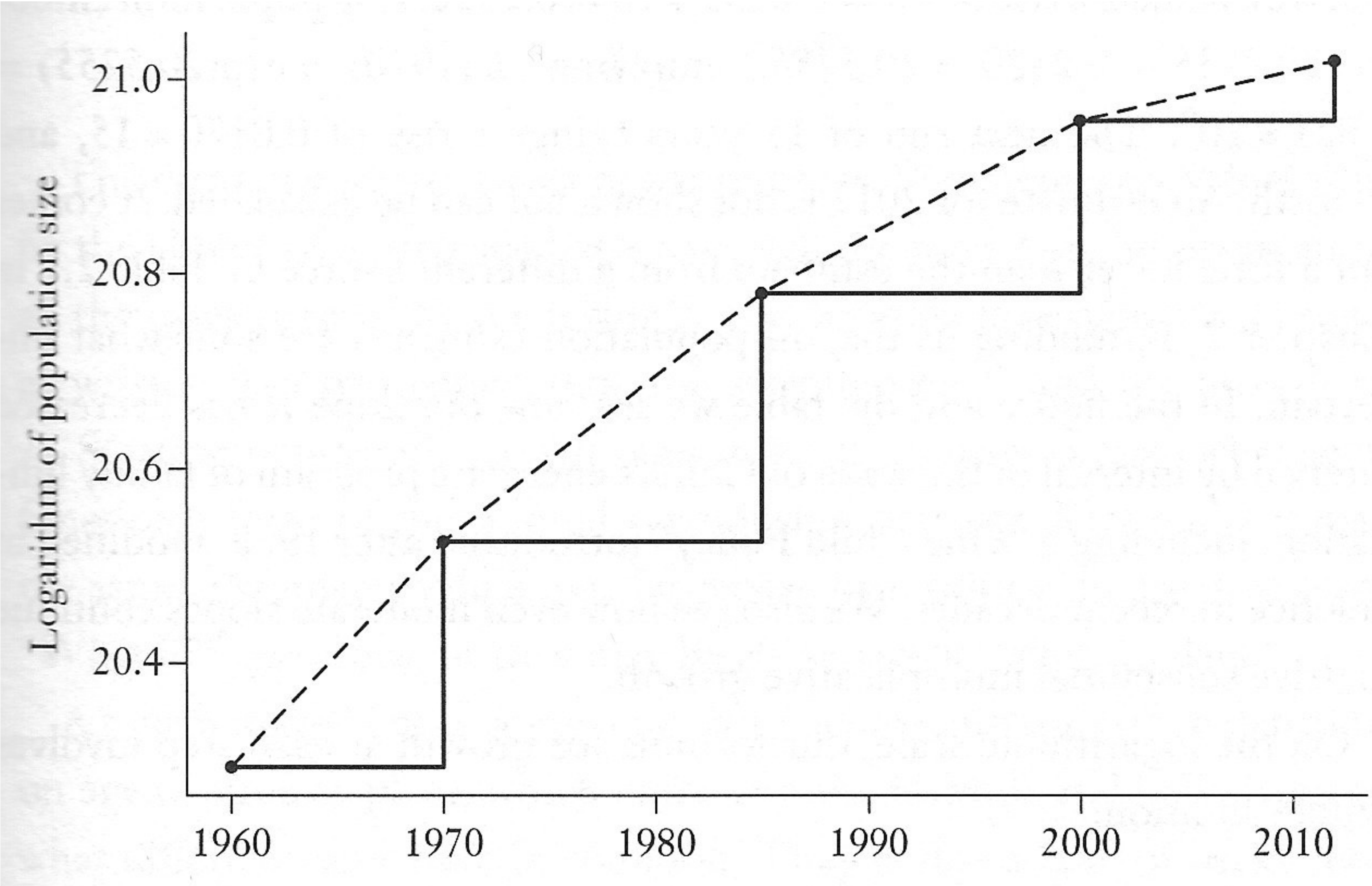
$$\log K(t+n) = \log K(t) + \mathrm{R}n$$

- We convert back to counts by applying the exponential function, which brings us back to multiplication

$$K(t+n) = K(t)\,\exp(\mathrm{R}n)$$

$$K(t+n) = K(t)\,e^{\mathrm{R}n}$$

Rise and run
China's log-population



Source: Wachter 2014, p. 15.

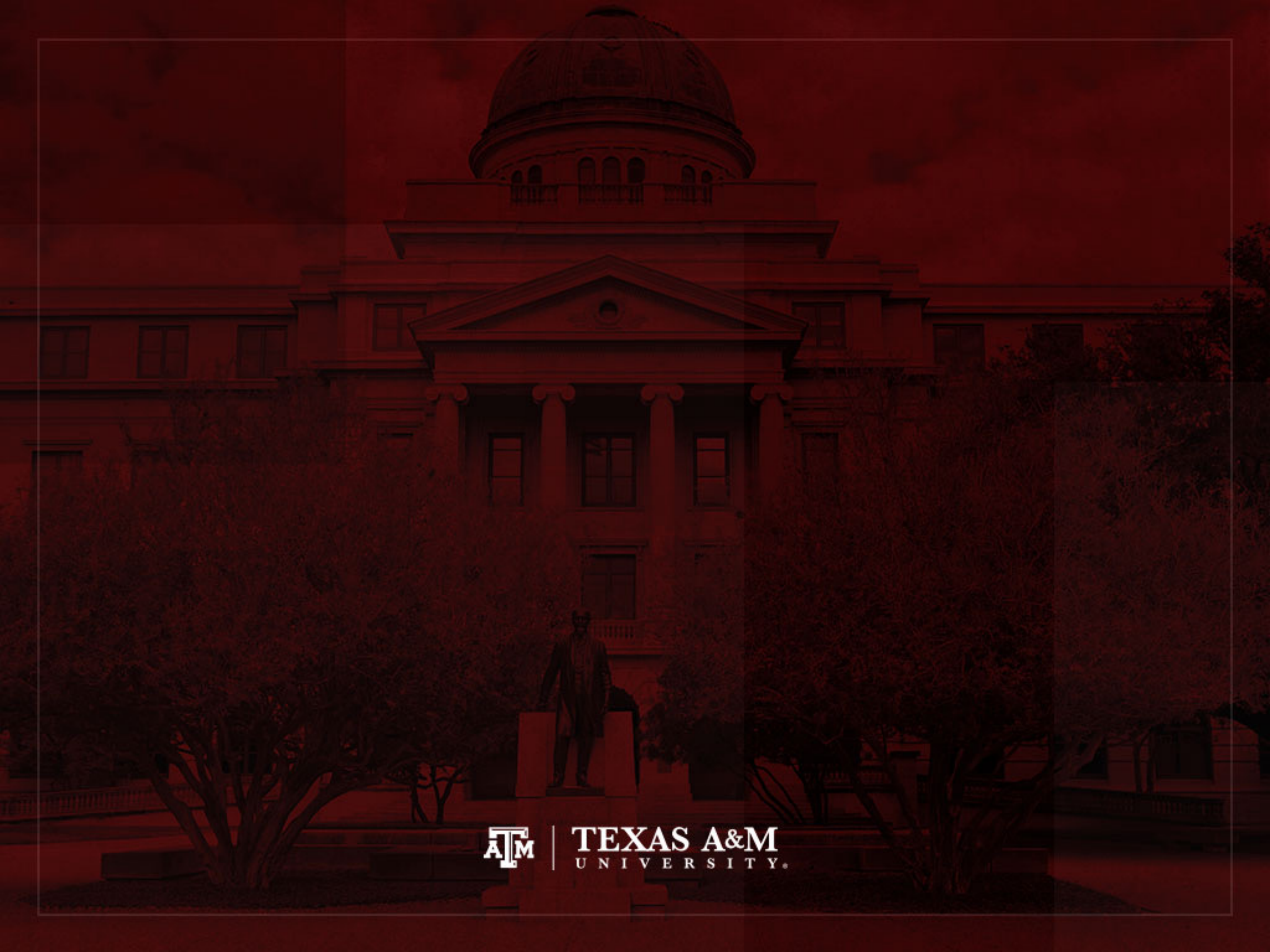
Growth rates in China

$$\log K(t+n) = \log K(t) + Rn$$

$$K(t+n) = K(t)\, e^{Rn} = \exp(\log(K))$$

Input log K(1960) = 20.2935			**Results**		
Date	*n* *"run"*	R	R*n* *"rise"*	log(K)	K(*t*)
1960	10	0.0232	0.2320	20.2935	650,661,438
1970	15	0.0170	0.2550	20.5255	820,561,976
1985	15	0.0117	0.1755	20.7805	1,058,903,738
2000	12	0.0052	0.0624	20.9560	1,262,045,936

Source: Census Bureau IDB (2012). Wachter 2014, p. 16.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Doubling times

- Doubling time: time it would take a population to double at a given growth rate if the exponential model were exactly true (rule of 69.3)

$$K(t) = \exp(\mathrm{R}t)\, K(0)$$

$$K(T_{\mathrm{double}}) = 2K(0) = \exp(\mathrm{R}T_{\mathrm{double}})\, K(0)$$

$$2 = \exp(\mathrm{R}T_{\mathrm{double}})$$

$$\log(2) = \mathrm{R}T_{\mathrm{double}}$$

$$T_{\mathrm{double}} = \log(2) / \mathrm{R} \approx 0.6931 / \mathrm{R}$$

- Halving time: if growth rate is negative, we would get how many years population would decrease by half

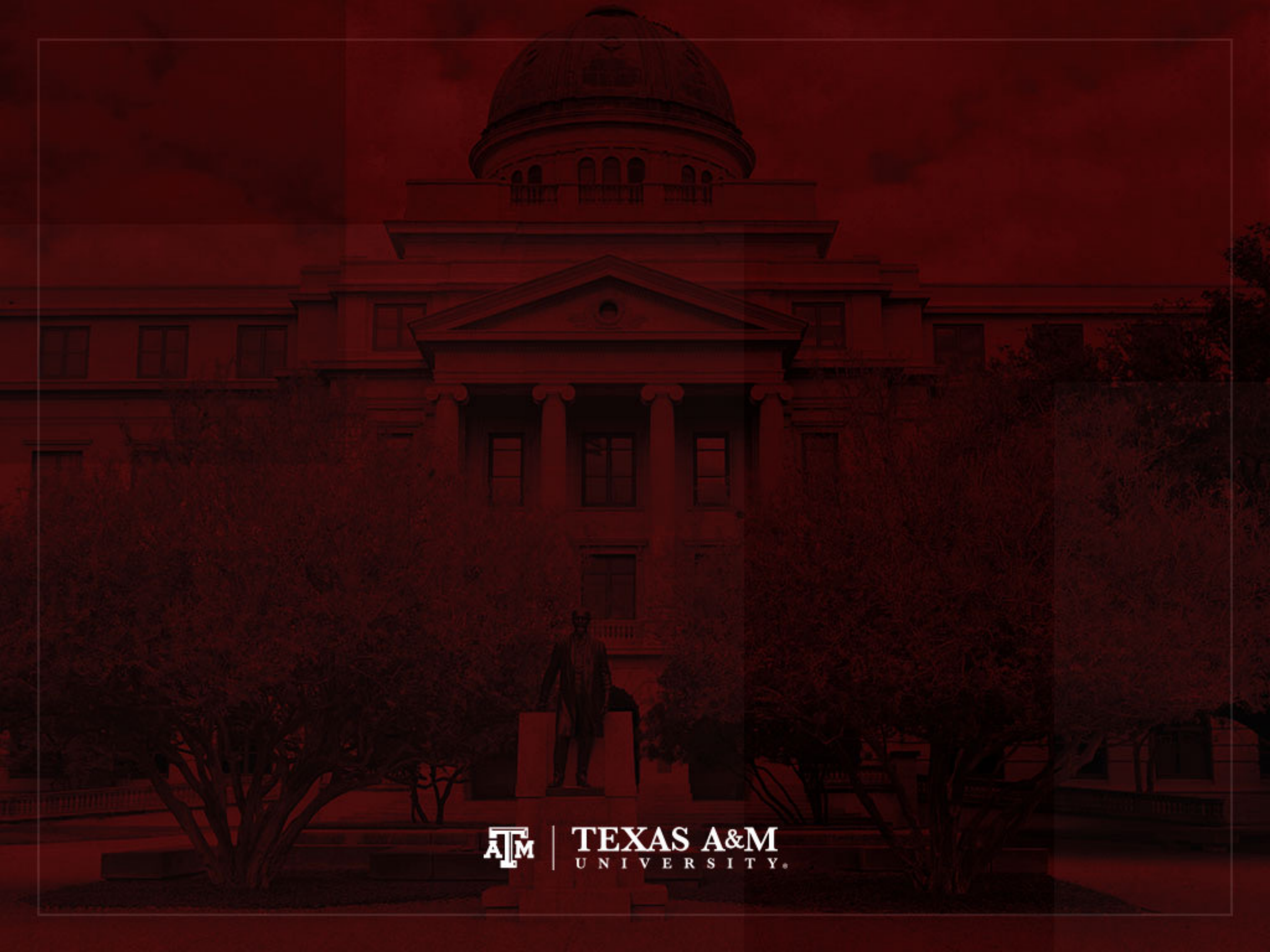
World population and doubling times

Date	Population	Growth rate (R)	Doubling time ≈ (0.6931 / R)
8000 B.C.	5 million	0.000489	1417 years
1 A.D.	250 million	–0.000373	–1858 years
600	200 million	0.000558	1272 years
1000	250 million	0.001465	473 years
1750	750 million	0.004426	157 years
1815	1,000 million	0.006957	100 years
1950	2,558 million	0.018753	37 years
1975	4,088 million	0.015937	43 years
2000	6,089 million		

Source: Estimates drawn from Cohen (1995) and IDB (2012). Wachter 2014, p. 25.

Reference

Wachter KW. 2014. Essential Demographic Methods. Cambridge: Harvard University Press. Chapter 1 (pp. 5–29).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