Lecture 3: Cohort mortality

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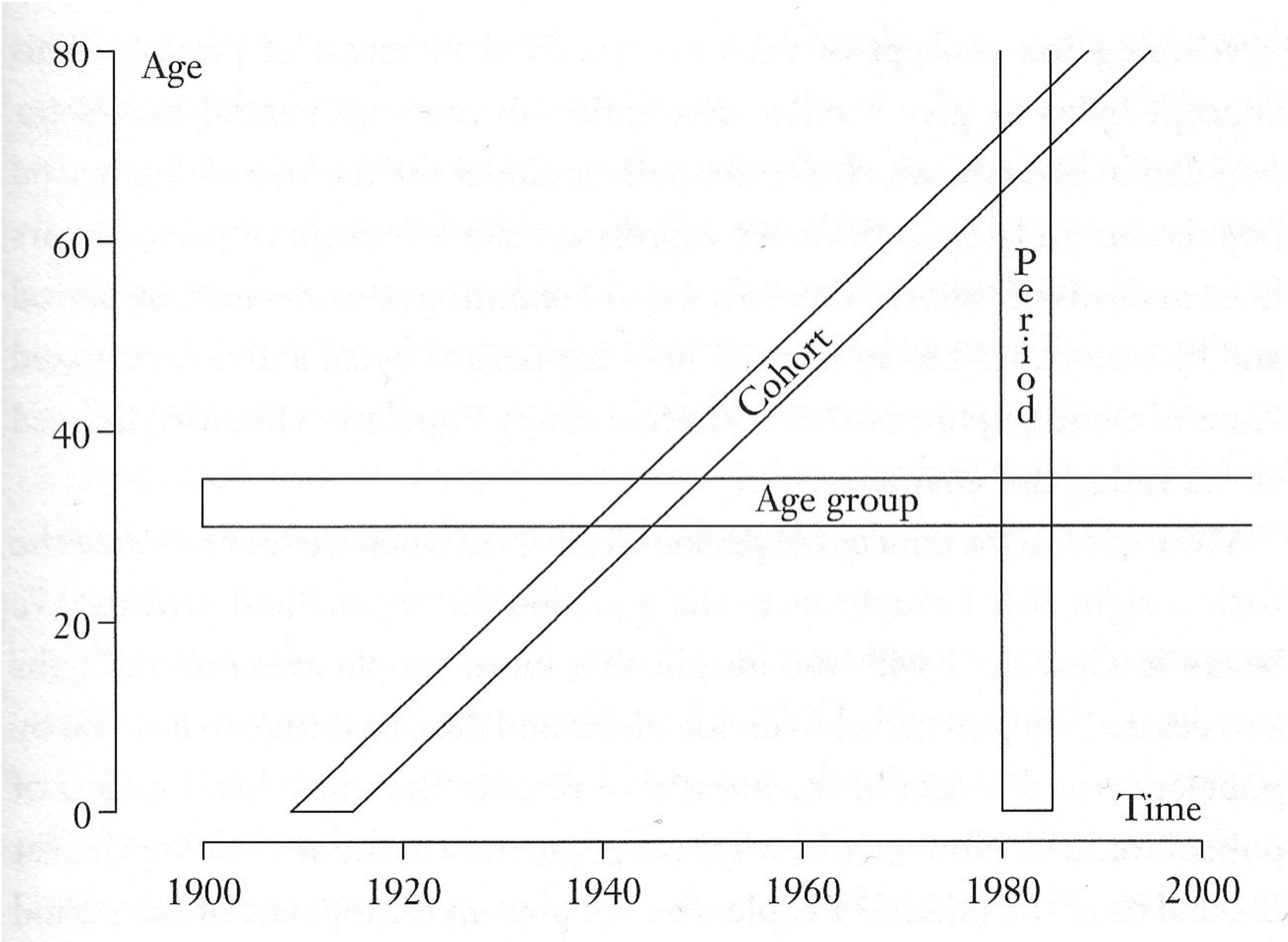
Cohort mortality

- Cohort survival by analogy
- Probabilities of dying
- Columns of the cohort life table
  - King Edward's children
  - From ${}_nL_x$ to e_x
  - Shapes of lifetable functions
  - The radix
- Annuities and insurance
- Mortality of the 1300s and 2000s

Cohort survival by analogy

- Analyze lifelines and the deaths that occur in the diagonal stripe on the Lexis diagram that represents a particular **cohort's** experience
  - Understand measures of survival and probabilities of dying as a function of age for the cohort
- We could also analyze the rectangle on a Lexis diagram that represents some **period**
  - We would consider the lifelines that cross the rectangle and the deaths that fall inside it

Lexis diagram



Source: Wachter 2014, p. 33.

Why start with cohort measures?

- The period measure is more complicated
  - People at risk of dying at different ages are different people
- For the experience of a cohort over time
  - People at risk of dying at different ages are the same people
- Cohort measures are conceptually simpler than period measures, so we begin with them

Disadvantage

- Disadvantage of cohorts measures is being out of date
- To have complete measures of cohort mortality for all ages, we have to wait until all members of the cohort have died
  - Rates for young ages refer to the distant past
- The most recent cohorts with complete mortality data are those born around 1900
- Measures of period mortality are more complicated, but they use more recent data

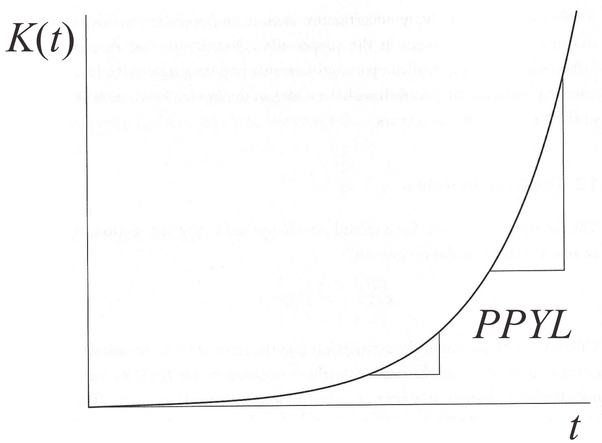
Basic cohort measures

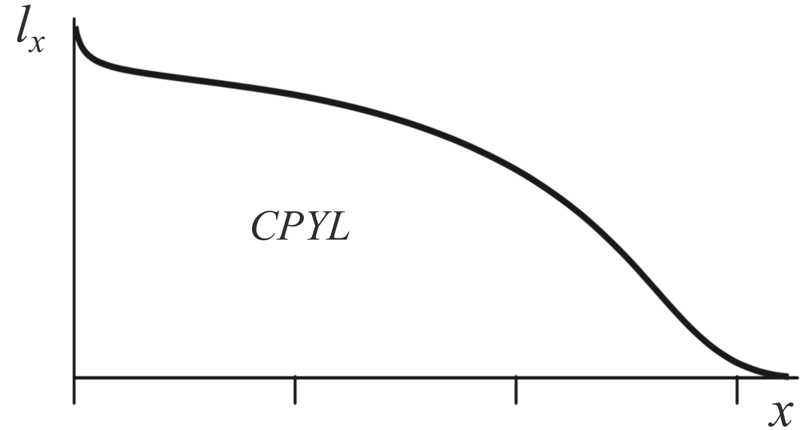
- The basic measures of cohort mortality are elementary
  - Take the model for exponential population growth
  - Apply it to a closed population consisting of the members of a single cohort
  - Change the symbols in the equations, but keep the equations themselves

Why is it a closed population?

- If our population consists of a single cohort
  - No one else enters the population after the cohort is born
  - Babies born to cohort members belong to later cohorts, not to their parents' cohort
  - For this cohort, the only changes in population size come from deaths to members of the population
- Measures from chapter 1 reappear with new names in an analogy between populations and cohorts...

Population Growth	Cohort Mortality
Time t	Age x
Population size $K(t)$	Cohort survivors ℓ_x
Multiplier $A = 1 - D/K$	Survival probability $1 - q = 1 - d / l$
Growth Rate R	Hazard rate h (with minus sign)
Area under $K(t)$, $PPYL$	Area under ℓ_x, $CPYL$, L
Crude rate d (function of t)	Age-specific rates m
Initial population $K(0)$	Initial cohort size l_0 (radix)
Population deaths D	Cohort deaths d





Note: A = 1 + (B – D)/K. But for a cohort, after age zero, births don't happen anymore.

Multiplication process

- The same process of multiplication for population growth happens for cohort mortality
  - It is not the mortality rates (*m*) that multiply
  - It is not the probabilities of dying (*q*)
  - It is the probabilities of surviving (1 – *q*)
- Age *x* is the subscript on the cohort survivors: l_x
  - Time *t* is used for population size: $K(t) = K_t$
- The notation is different but the idea is the same

Multiplicative rules

- Multiplicative rule for population growth

$$K(t + n) = A\, K(t)$$

- Multiplicative rule for cohort survivorship

$$l_{x+n} = (1 - {}_nq_x)\, l_x$$

  - Subscript ***n*** specifies the length of the interval
  - ${}_n\boldsymbol{q}_x$: probability of dying within an interval of length n that starts at age x and ends at age $x+n$
  - $\boldsymbol{l}_{x+n}$: members who survive to age $x+n$

More notations

- ***$_nq_x$*** : **probability of dying** between ages x and $x+n$ among cohort members alive at age x

$$_nq_x = {_nd_x} / l_x$$

- **1 − *$_nq_x$*** : **probability of surviving** from age x to age $x+n$ among cohort members alive at age x

$$1 - {_nq_x} = l_{x+n} / l_x$$

- ***$_nd_x$*** : cohort deaths between ages x and $x+n$
- ***$_nL_x$*** : cohort person-years lived in this interval
- ***l_x*** : cohort members alive at age x are split in two
  - ***$_nd_x$*** : members who die before age $x+n$
  - ***l_{x+n}*** $= l_x - {_nd_x}$: members who survive to age $x+n$

Some more notations

- In the expression $_{n}q_{x}$ the left subscript gives the width of the age interval and the right subscript gives the starting age
- $_{10}q_{20}$: probability of dying between 20 and 30
  - Not between 10 and 20
  - Do not confuse $_{n}q_{x}$ with n multiplied by q_x
  - If you want to multiply n by l_x, use this notation: $(n)(l_x)$
- $_{10}q_{20}$ goes from 20.00000 to 29.99999
  - "The interval from 20 to 30" (including exact age 20, excluding exact age 30)
  - Some authors call it "the interval from 20 to 29"

Example

- A cohort born in 1984 reached age 18 in 2002 and 1,767,644 were alive at their 18th birthday
  - Only 724 of them died before age 19

- Probability of dying

$$ {}_1q_{18} = {}_1d_{18} / l_{18} = 724 / 1{,}767{,}644 = 0.000410 $$

- Probability of surviving

$$ 1 - {}_1q_{18} = 1 - 0.000410 = 0.999590 $$

$$ l_{19} / l_{18} = 1{,}766{,}920 / 1{,}767{,}644 = 0.999590 $$

Hazard rates

- Hazard rates can express the pace of death within cohorts
- Hazard rate is the counterpart of population growth rate
  - We measure population growth with slopes of logarithms of population size
  - We can measure cohort losses with slopes of logarithms of numbers of survivors
- We insert a minus sign to make the hazard rate into a positive number
  - Because cohorts decrease as they age
  - i.e., cohorts grow smaller, not larger, as they age

Hazard rate formula

- The hazard rate for a cohort is minus the slope of the logarithm of the number of cohort survivors as a function of age
- Expressing hazard rate (h_x) in the interval starting at age x (omitting any subscript for n)

$$h_x = -\frac{1}{n} log\left(\frac{l_{x+n}}{l_x}\right) \qquad R = \frac{1}{n} log\left(\frac{K_{t+n}}{K_t}\right)$$

- Formula for cohort survivorship resemble formula for exponential population growth

$$l_{x+n} = l_x\, e^{-nh_x} = l_x\, e^{-h_x n}$$

$$K_{t+n} = K_t\, e^{Rn}$$

Example

- The cohort of boys born in the United States in 1980 started out with 1,853,616 members
  - 1,836,853 of them survived to their first birthday

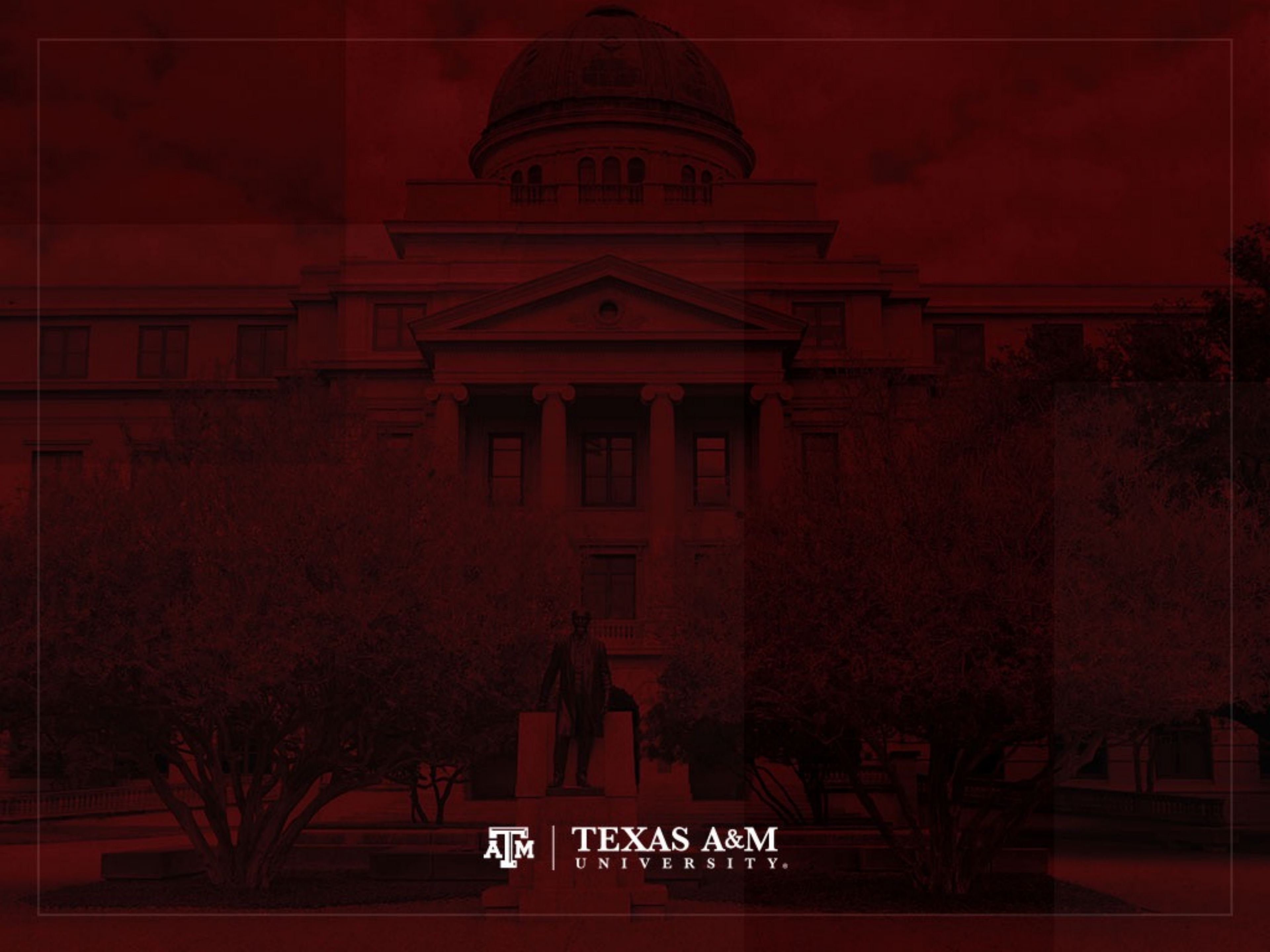
$$h_x = -\frac{1}{n} log\left(\frac{l_{x+n}}{l_x}\right)$$

$$h_x = -\frac{1}{1} log\left(\frac{1{,}836{,}853}{1{,}853{,}616}\right)$$

$$h_x = -\ log(0.990957)$$

$$h_x = -(-0.009084)$$

$$h_x = 0.009084$$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Probabilities of dying

- A hazard rate is a rate like *R*, whereas ${}_nq_x$ is a probability
- The word "probability" suggests a random process
- Randomness refers in principle to a randomly selected member of our cohort
- The occurrence of death appears partly random and partly determined by causes
  - These causes are partly random and partly determined by prior causes

${}_nq_x$-conversions

- Problems that involve working out ${}_nq_x$ values for different x and n are called "${}_nq_x$-conversions"
- Demographers frequently find themselves with data for one set of age intervals when they need answers for different intervals
  - They may have data for 1-year-wide intervals and need answers for 5-year-wide intervals
  - They may have data for 15-year intervals and need answers for 5-year intervals
  - They may have tables for ages 25 and 30 and need to know how many women survive to a mean age of childbearing of, for example, 27.89 years

Applying multiplication to l_x

- From our analogy with population growth, we go from l_x to l_{x+n} by multiplication
- We go from l_{65} to l_{85} by multiplying by $1 - {}_{20}q_{65}$

$$l_{85} = (1 - {}_{20}q_{65})\, l_{65}$$

- We go from l_{85} to l_{100} by multiplying by $1 - {}_{15}q_{85}$

$$l_{100} = (1 - {}_{15}q_{85})\, l_{85}$$

- We go from l_{65} to l_{100} by multiplying by the product $(1 - {}_{20}q_{65})(1 - {}_{15}q_{85})$

$$l_{100} = (1 - {}_{20}q_{65})(1 - {}_{15}q_{85})\, l_{65}$$

Survival probabilities multiply

- While we are interested in q, we work with $1 - q$
- We do not multiply the ${}_nq_x$ values
  - To die, you can die in the first year *or* in the second year *or* in the third year, and so on
  - You only do it once
  - There is no multiplication
- We multiply the $1 - {}_nq_x$ values
  - To survive 10 years you must survive the first year *and* survive the second year *and* survive the third year, and so on
  - These "ands" mean multiplication

Basic assumption

- We need an assumption when we **do not have** direct data for **short intervals** of interest, such as 1-year-wide intervals
- We need an assumption when we **only have** data for **wider intervals**, such as 5-year-wide intervals
- We assume the probability of dying is constant within each interval where we have no further information

Applying assumption

- If we do not know ${}_1q_{20}$ or ${}_1q_{21}$ but we do know ${}_2q_{20}$
- We assume that the probability of dying is constant between ages 20 and 22

$${}_1q_{20} = {}_1q_{21} = q$$

- Then $(1 – q)^2$ has to equal $1 – {}_2q_{20}$
- More generally, for y between x and $x+n–1$

$$(1 – {}_1q_y)^n = 1 – {}_nq_x$$

$$1 - {}_1q_y = (1 – {}_nq_x)^{1/n}$$

$${}_1q_y = 1 - (1 - {}_nq_x)^{1/n}$$

Example 1

- For the cohort of U.S. women born in 1980

$$_{2}q_{20} = 0.000837$$

- Calculate $_{1}q_{20}$

$_{1}q_{y} = 1 - (1 - {}_{n}q_{x})^{1/n}$

$_{1}q_{20} = 1 - (1 - {}_{2}q_{20})^{1/2} = 1 – (1 – 0.000837)^{1/2}$

$_{1}q_{20} = 1 – (0.999163)^{1/2} = 1 – 0.999581$

$_{1}q_{20} = 0.000419$

Example 2

- For the cohort of women born in 1780 in Sweden

$${}_5q_{20} = 0.032545$$

- Calculate ${}_1q_{20}$

${}_1q_y = 1 - (1 - {}_nq_x)^{1/n}$

${}_1q_{20} = 1 - (1 - {}_5q_{20})^{1/5} = 1 – (1 – 0.032545)^{1/5}$

${}_1q_{20} = 1 – (0.967455)^{1/5} = 1 – 0.993405$

${}_1q_{20} = 0.006595$

Example 3

- Suppose we know that ${}_5q_{80} = 0.274248$
  - We want to find the probability of dying each year which would, if constant, account for the observed 5-year mortality and survivorship
- Calculate ${}_1q_{80}$

$${}_1q_y = 1 - (1 - {}_nq_x)^{1/n}$$

$${}_1q_{80} = 1 - (1 - {}_5q_{80})^{1/5} = 1 - (1 - 0.274248)^{1/5}$$

$${}_1q_{80} = 1 - (0.725752)^{1/5} = 1 - 0.937902$$

$${}_1q_{80} = 0.062098$$

Example 4

- More elaborate conversion problems arise
- We might have values from a forecast of survival for the U.S. cohort of women born in 1980

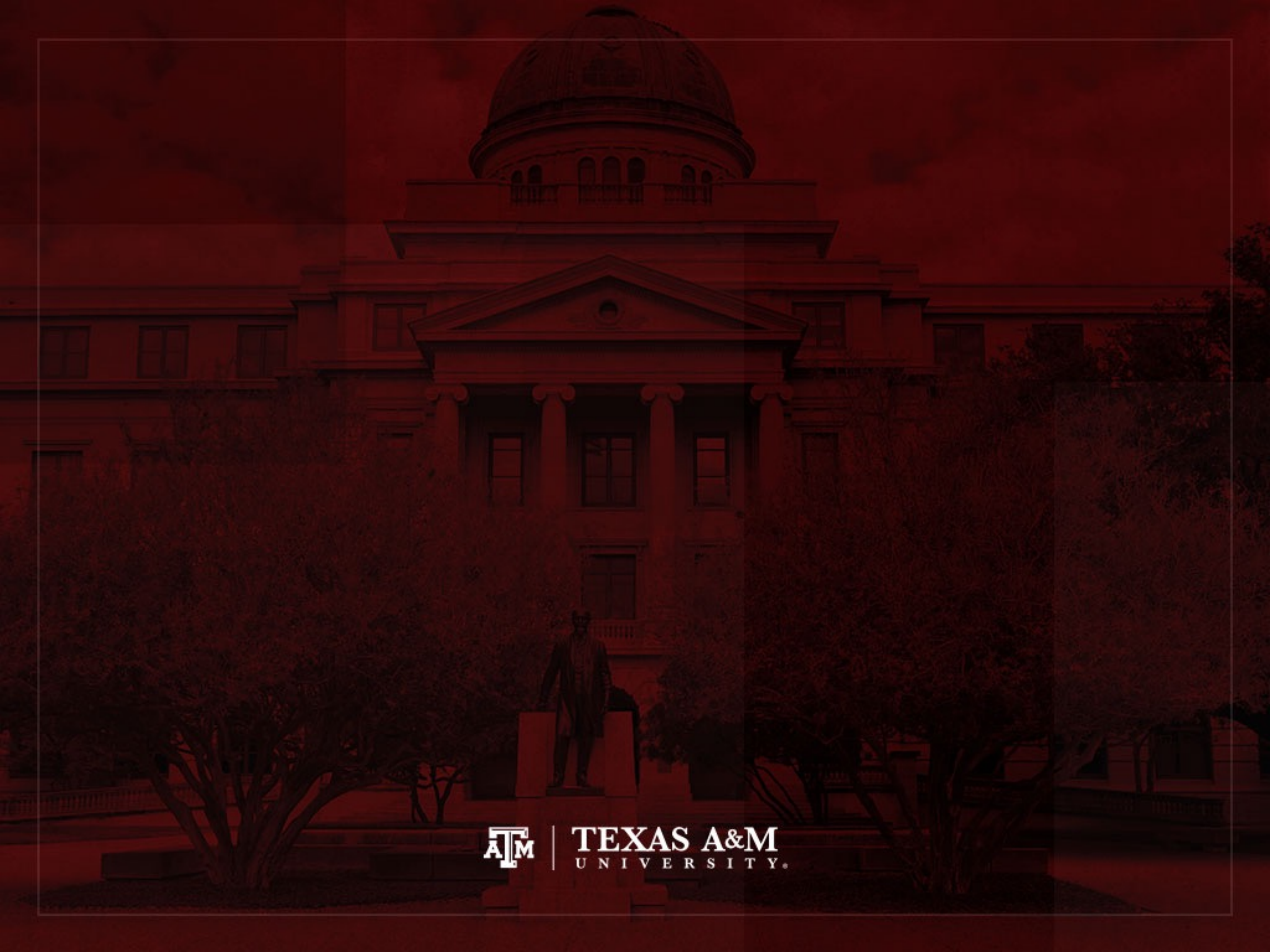
  $l_{65} = 0.915449$; $l_{75} = 0.799403$; ${}_{35}q_{65} = 0.930201$
- We might want the probability of surviving from 70 to 100: l_{100} / l_{70}

$$\frac{l_{100}}{l_{70}} = 1 - {}_{30}q_{70} = \frac{\frac{l_{100}}{l_{65}}}{\frac{l_{70}}{l_{65}}} = \frac{1 - {}_{35}q_{65}}{1 - {}_{5}q_{65}} = \frac{1 - 0.930201}{(1 - {}_{10}q_{65})^{5/10}} = \frac{0.069799}{(1 - {}_{10}q_{65})^{1/2}} = \frac{0.069799}{(l_{75}/l_{65})^{1/2}}$$

$$\frac{l_{100}}{l_{70}} = \frac{0.069799}{\left(\frac{0.799403}{0.915449}\right)^{\frac{1}{2}}} = \frac{0.069799}{(0.873236)^{\frac{1}{2}}} = \frac{0.069799}{0.934471} = 0.074694$$

Use the Lexis diagram

- The best way to solve complicated conversion problems is to begin by drawing a diagonal line on a Lexis diagram
  - Mark off each age for which there is information about survivorship at that age
  - Mark off ages which are the endpoints of intervals over which there is information about mortality within the interval
  - Between each marked age, assume a constant probability of dying, and apply the conversion formulas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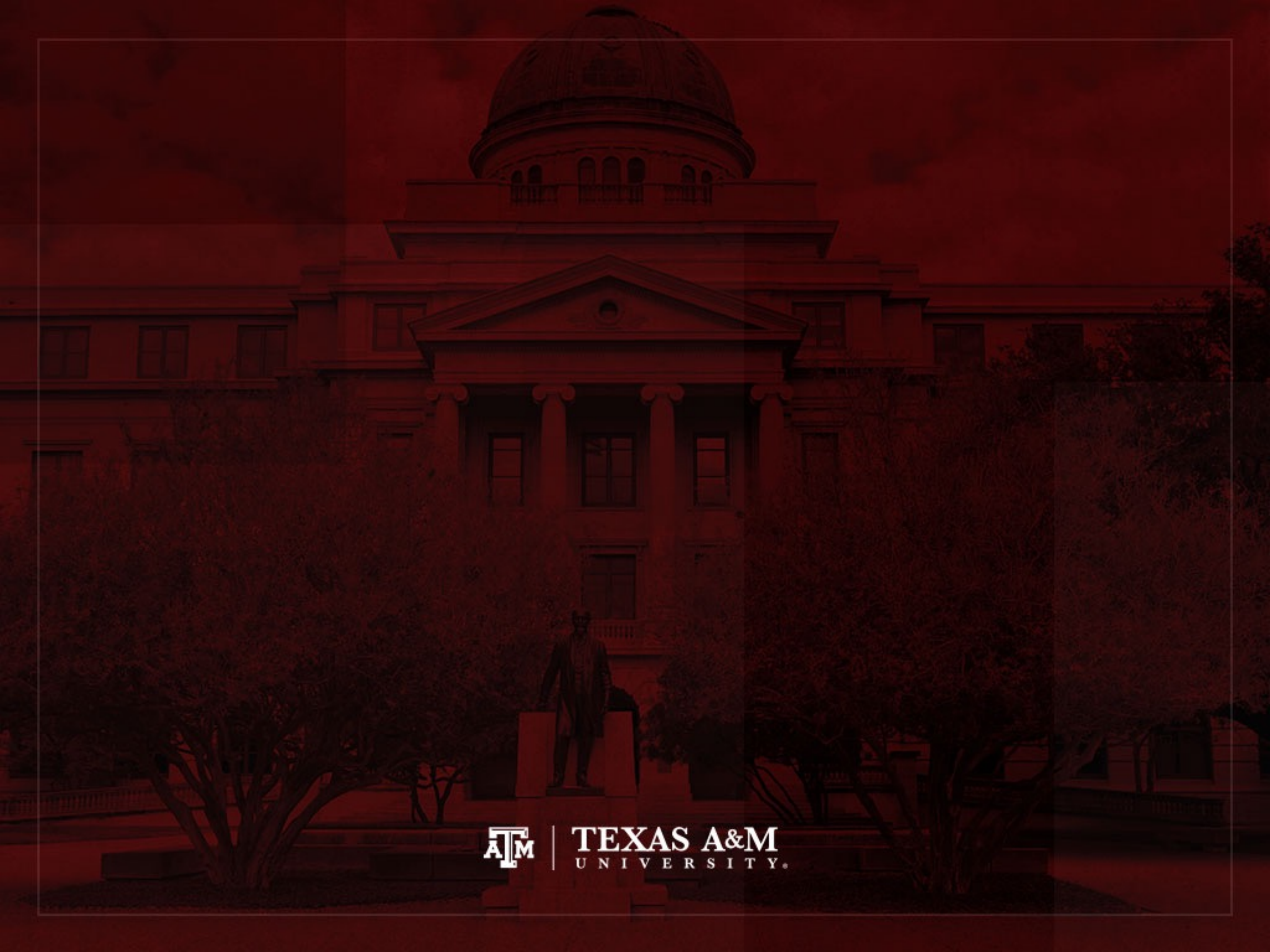
Columns of the cohort life table

- Lifetable is a table with l_x and ${}_nq_x$ as columns with a set of other measures of mortality
- Columns and their names and symbols are fixed by tradition
  - This is customary since the 1600s
  - Each column is a function of age, so the columns of the lifetable are sometimes called “lifetable functions”
- Rows correspond to age groups

Information in lifetable columns

- All the main columns of the lifetable contain the same information from a mathematical point of view
  - With some standard assumptions any column can be computed from any other
- But they present information from different perspectives for use in different applications
  - Survivors
  - Deaths
  - Average life remaining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King Edward's children

- King Edward III of England was born in 1312 and reigned from 1337 to 1377

Table 3.2 Children of King Edward III of England

1330–1376	Edward, The Black Prince
1332–1382	Isabel
1335–1348	Joan
1336–?	William of Hatfield (died young)
1338–1368	Lionel of Antwerp, Duke of Clarence
1340–1398	John of Gaunt, Duke of Lancaster
1341–1402	Edmund Langley, Duke of York
1342–1342	Blanche
1344–1362	Mary
1346–1361	Margaret
1355–1397	Thomas of Woodstock, Duke of Gloucester

Source: Wachter 2014, p. 55.

Lexis diagram for King Edward's children

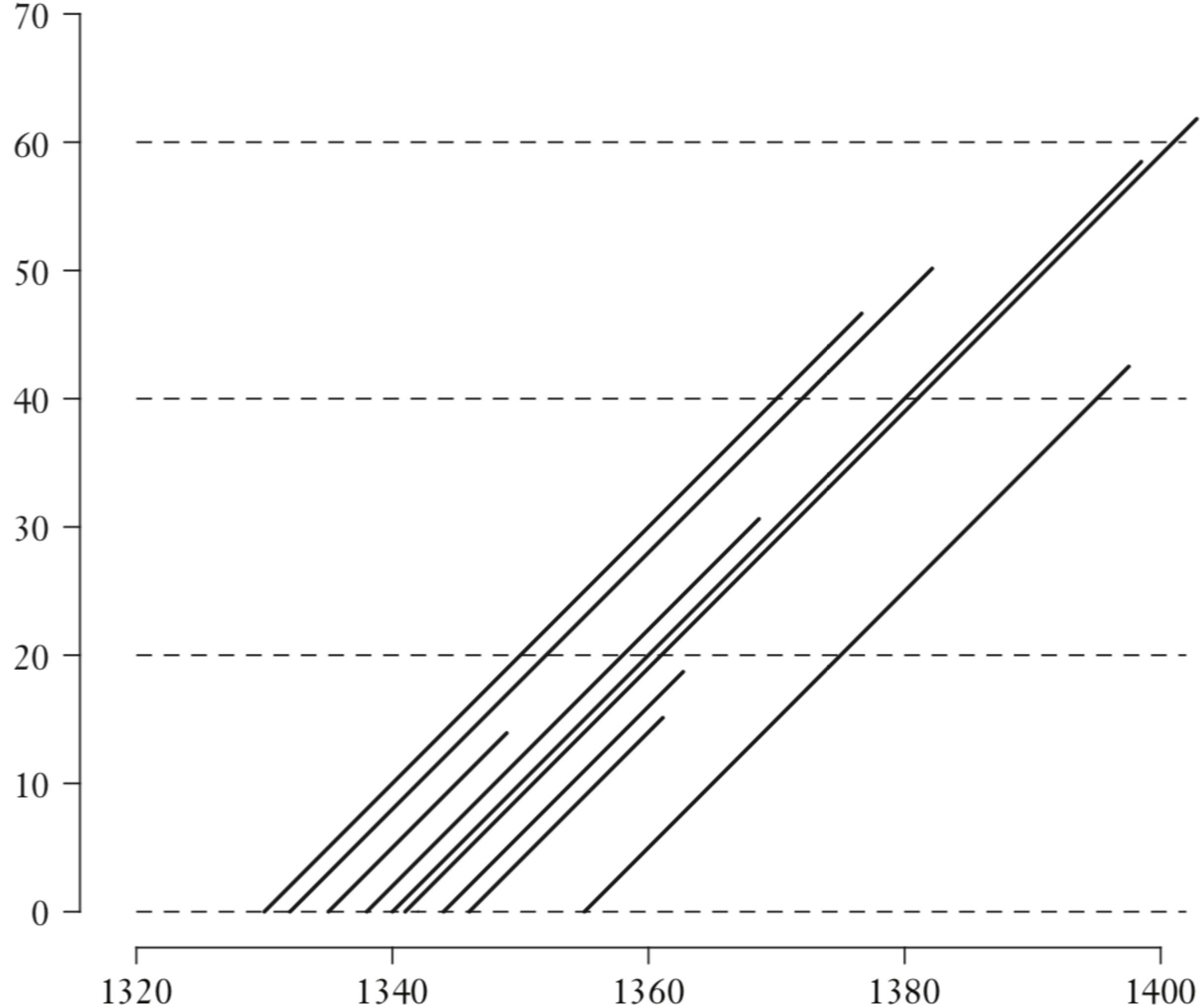


Figure 3.1 Lexis diagram for the children of King Edward III

Source: Wachter 2014, p. 56.

Constructing a cohort life table

- Generally, lifetables are constructed with 1-year or 5-year intervals
  - A complete life table provides life table functions in single years of age
  - Lifetables in which functions are given for age groups are called “abridged lifetables” in older works
- Usually, for lifetables with 5-year-wide intervals
  - The first age group is a 1-year-wide interval (0–1)
  - The second age group is a 4-year-wide interval (1–5)

Table 3.3 Five columns of King Edward's family lifetable

x	n	ℓ_x	${}_nq_x$	${}_nd_x$
0	10	10	0.100	1
10	10	9	0.333	3
20	20	6	0.167	1
40	20	5	0.800	4
60	∞	1	1.000	1

Source: Wachter 2014, p. 58.

Start of age group (x) and width (n)

- Lifetables begin with a column labeled x
  - Starting age for the age group
- The next column has the width n of the age group
  - The difference between the value of x for this row and the value for the next age group found in the next row
- The last age group is called the “open-ended age interval” since it has no maximum age
  - Symbol for infinity (∞) is used for the length of this interval
  - We don’t set any upper limit of our own

Number of survivors (l_x)

- The survivorship column l_x leads off the data-driven entries of the cohort lifetable

$$l_{x+n} = l_x\,(1 - {}_nq_x)$$

- The first-row entry (l_0) is the radix, the initial size of the cohort at birth
  - The choice of radix is up to us
  - A lifetable can be built up from any radix, an actual size or a convenient size

Graph of l_x as a function of x

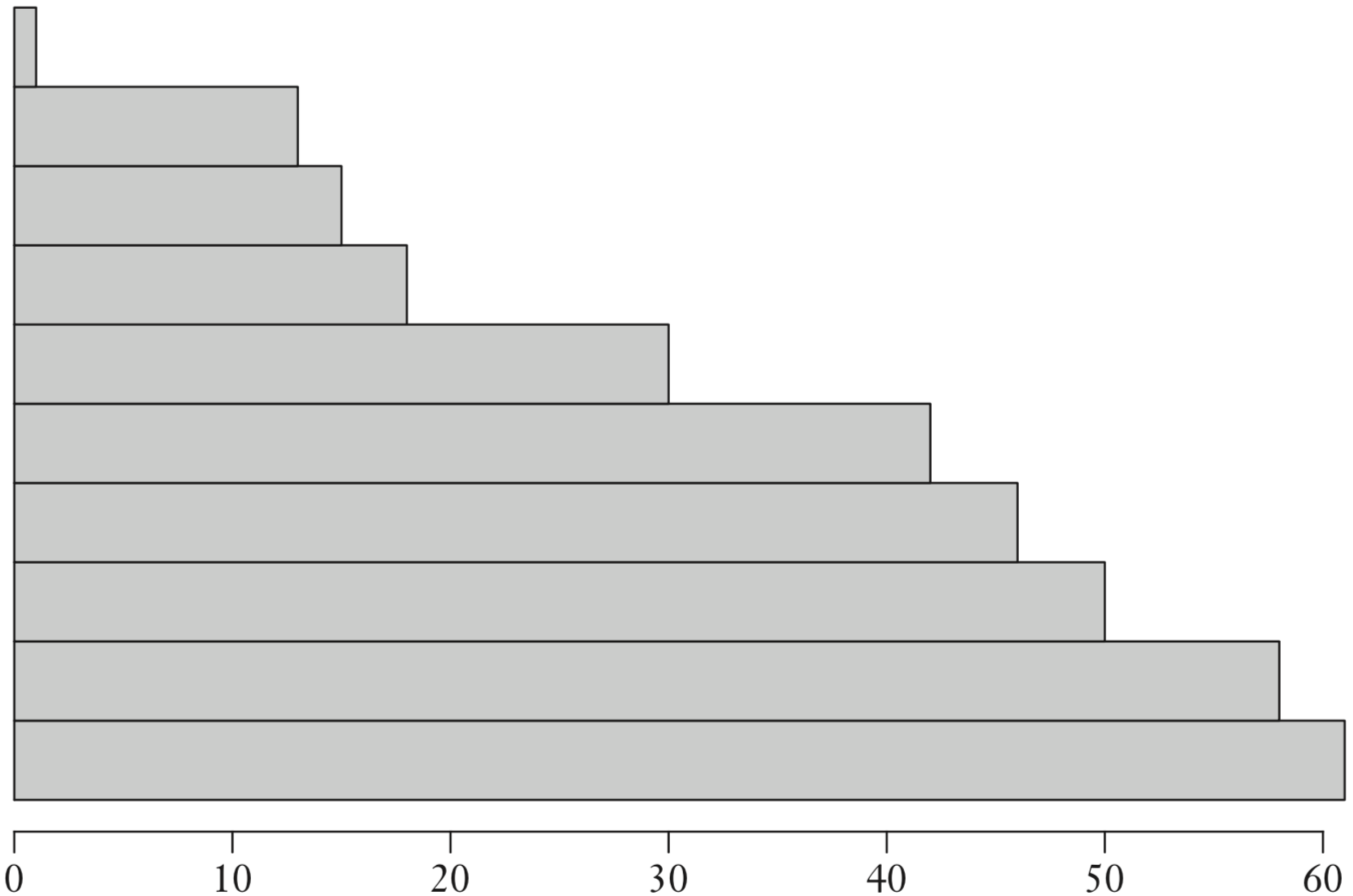


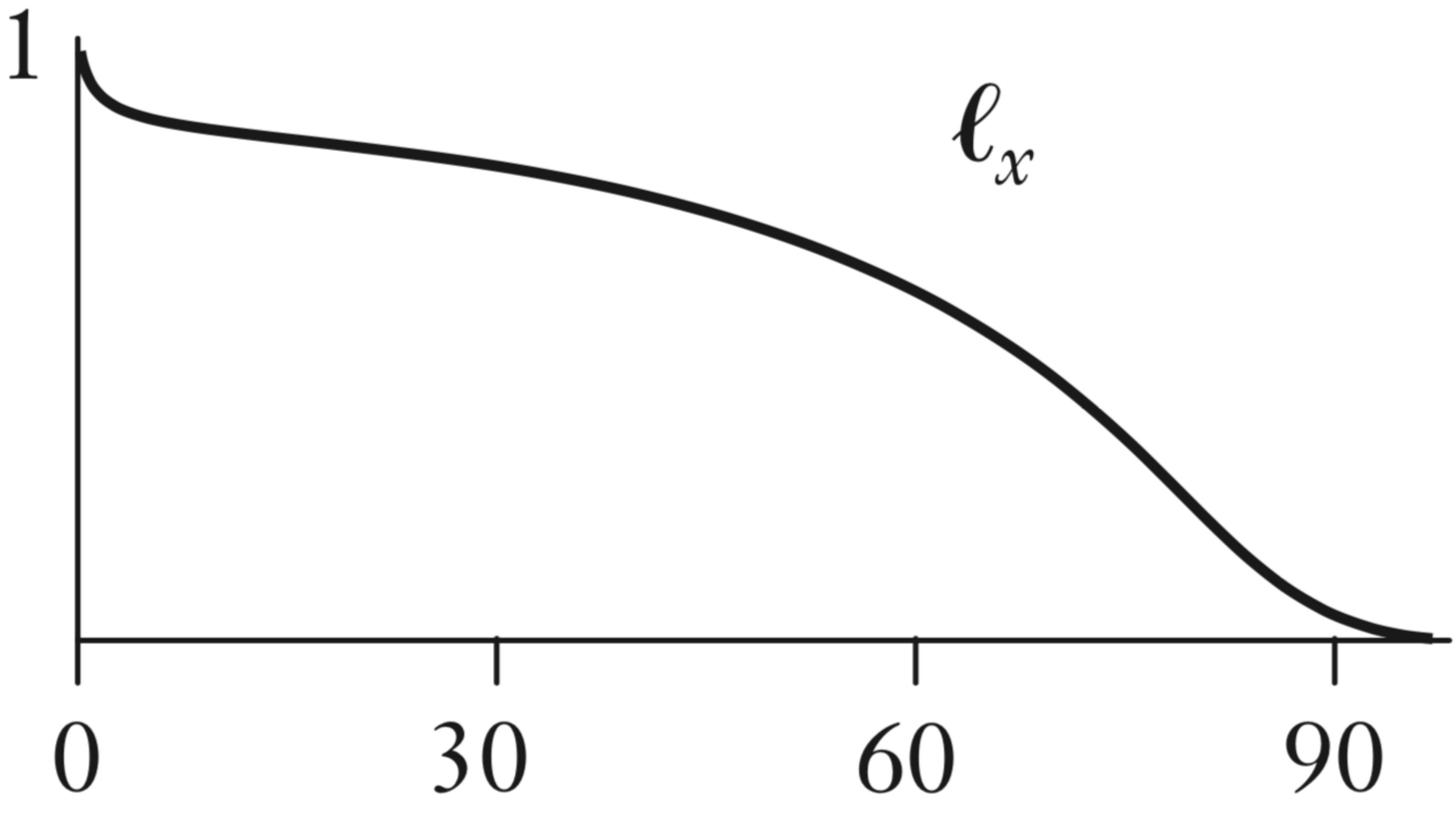
Figure 3.2 Lifespans and ℓ_x

Source: Wachter 2014, p. 57.

Continuous function for l_x

- The previous plot for a large population has lots of very thin bars
- We often draw a smooth curve through the mid-points of the right-hand sides of the bars
  - Instead of taking steps down, l_x becomes a continuous function
- Demographers often draw the bars with different colors for the portions of each person's life spent in and out of some activity
  - Rearing children, being married, free from disability

Typical shape of l_x



Source: Wachter 2014, p. 61.

Probability of dying (${}_nq_x$)

- The column which follows l_x in the lifetable contains the probability of dying in the interval given that one is alive at the start
- This is the ${}_nq_x$ measure

$${}_nq_x = 1 - (l_{x+n} / l_x)$$

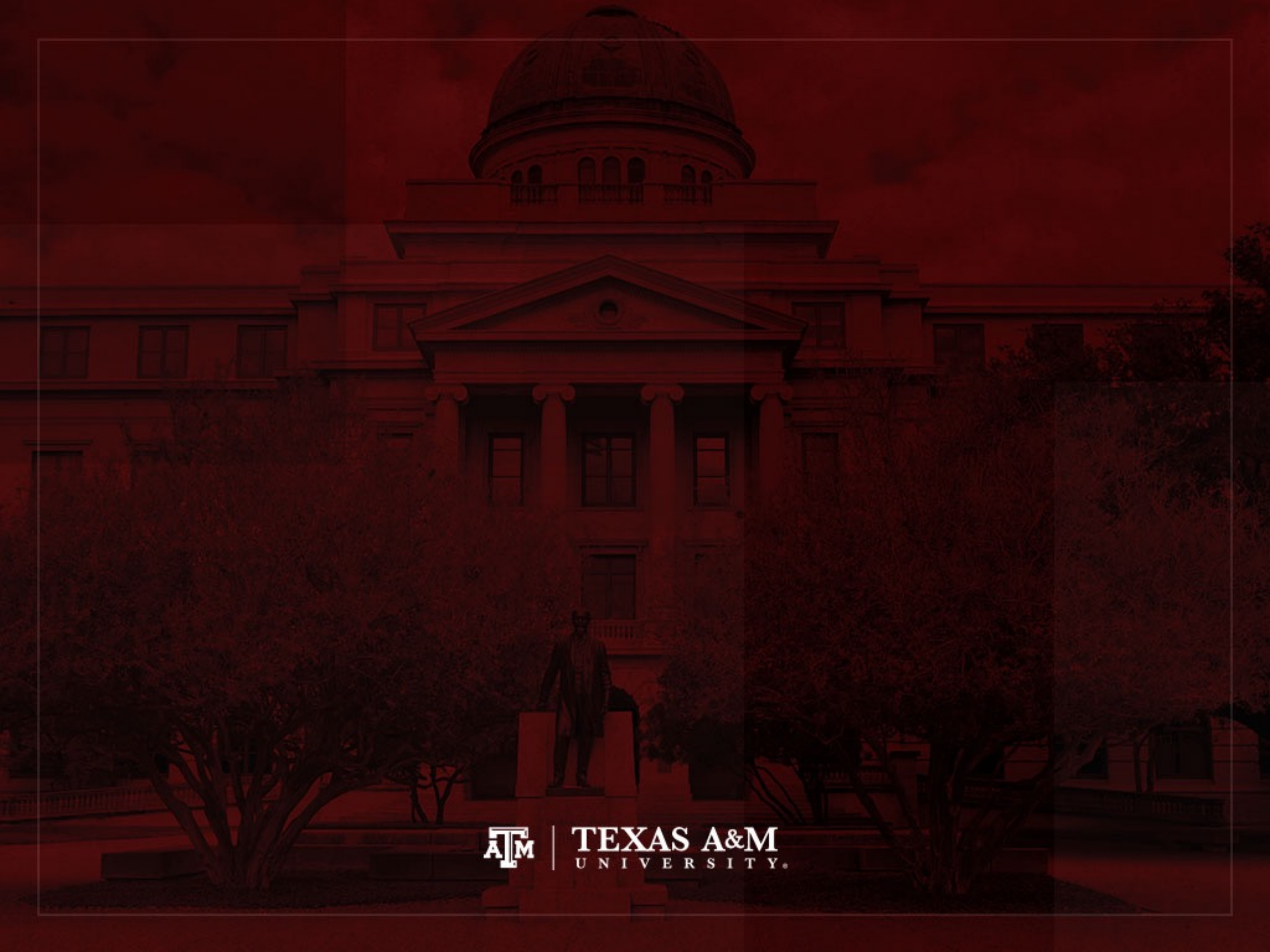
- For our example
  - In the first age group, ${}_nq_x = 1 - 9/10 = 0.100$
  - In the second age group, ${}_nq_x = 1 - 6/9 = 0.333$

Number of deaths (${}_nd_x$)

- We go on to insert a column which gives deaths between ages x and x + n

$$ {}_nd_x = l_x - l_{x+n} $$

- This column counts the lifelines that end in each age interval on the Lexis diagram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From ${}_nL_x$ to e_x

- The remaining columns of the lifetable relate to cohort person-years lived (CPYL)
- In order to calculate person-years, we need ${}_na_x$
- ${}_na_x$ tells us how many years within an interval people live on average if they die in the interval
- This quantity is about half the width of the interval (n/2)

Table 3.4 Right-hand columns of a lifetable

x	${}_na_x$	${}_nL_x$	${}_nm_x$	T_x	e_x	$x + e_x$
0	0.50	90.5	0.011	334	33.4	33.4
10	5.33	76.0	0.039	243	27.0	37.0
20	10.00	110.0	0.009	167	27.8	47.8
40	9.00	56.0	0.071	57	11.4	51.4
60	1.00	1.0	1.000	1	1.0	61.0

Source: Wachter 2014, p. 60.

Cohort person-years lived (${}_nL_x$)

- With ${}_na_x$, we can calculate cohort person-years lived between ages x and $x+n$ (${}_nL_x$)
  - Also called "big *L*"
  - Think of "*L*" standing for life
- Big *L* is one of the four most important columns with
  - l_x , "little *l*"
  - ${}_nq_x$
  - e_x

Formula of ${}_nL_x$

- The value of ${}_nL_x$ is made up of two contributions
  - Those who survive the whole interval (l_{x+n}) contribute a full n years to ${}_nL_x$
  - Those who die during the interval (${}_nd_x$) contribute on average ${}_na_x$ years
- Our formula adds these two contributions

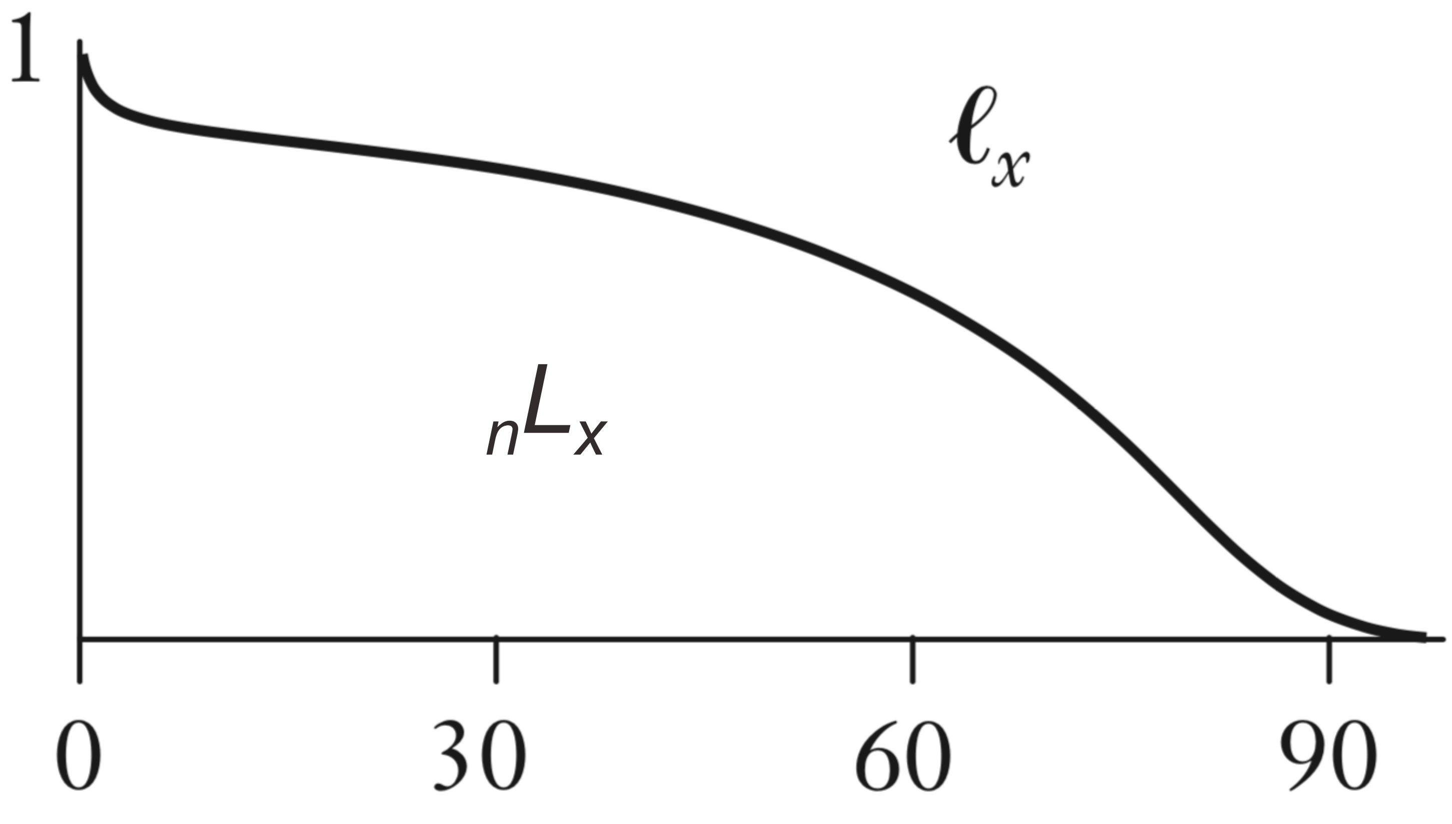
$${}_nL_x = (n)\,(l_{x+n}) + ({}_na_x)\,({}_nd_x)$$

- We usually have ${}_na_x = n/2$, then formula simplifies

$${}_nL_x \approx (n/2)\,(l_x + l_{x+n})$$

${}_nL_x$ as the area under the l_x curve

- With a smooth curve of l_x, we can calculate ${}_nL_x$ as the area under the l_x curve between x and $x+n$



Source: Wachter 2014, p. 61.

Death rate (${}_n m_x$)

- For the lifetable death rate (${}_n m_x$), we divide cohort deaths (${}_n d_x$) by cohort person-years lived (${}_n L_x$)
- The column ${}_n m_x$ is the age-specific counterpart of the crude death rate (CDR)
- ${}_n m_x$ is a rate, measured per unit of time
- The lifetable death rate measured over a very short interval starting at x is very close to the hazard rate

Remaining person-years of life (T_x)

- We can obtain person-years of life remaining for cohort members who reach age x (T_x)

- We simply add up all person-years to be lived beyond age x

- T_x is easiest to compute by filling the whole ${}_nL_x$ column and cumulating sums from the bottom up

$$T_x = {}_nL_x + {}_nL_{x+n} + {}_nL_{x+2n} + \ldots$$

Remaining life expectancy (e_x)

- The main use of T_x is for computing the expectation of further life beyond age x (e_x)
- The T_x person-years will be lived by the l_x members of the cohort who reach age x
  - So, e_x is given by the formula

$$e_x = T_x / l_x$$

- The expectation of life at age zero (at birth) is often called the life expectancy (e_0)

Average age at death ($x + e_x$)

- e_x is the expectation of future life beyond age x
  - It is not an average age at death
- We add x and e_x to obtain the average age at death for cohort members who survive to age x
  - Not all lifetables include $x + e_x$
  - The $x + e_x$ column always go up
- e_x does not always go down
  - It often goes up after the first few years of life, because babies who survive infancy are no longer subject to the high risks of infancy

Index of lifespan

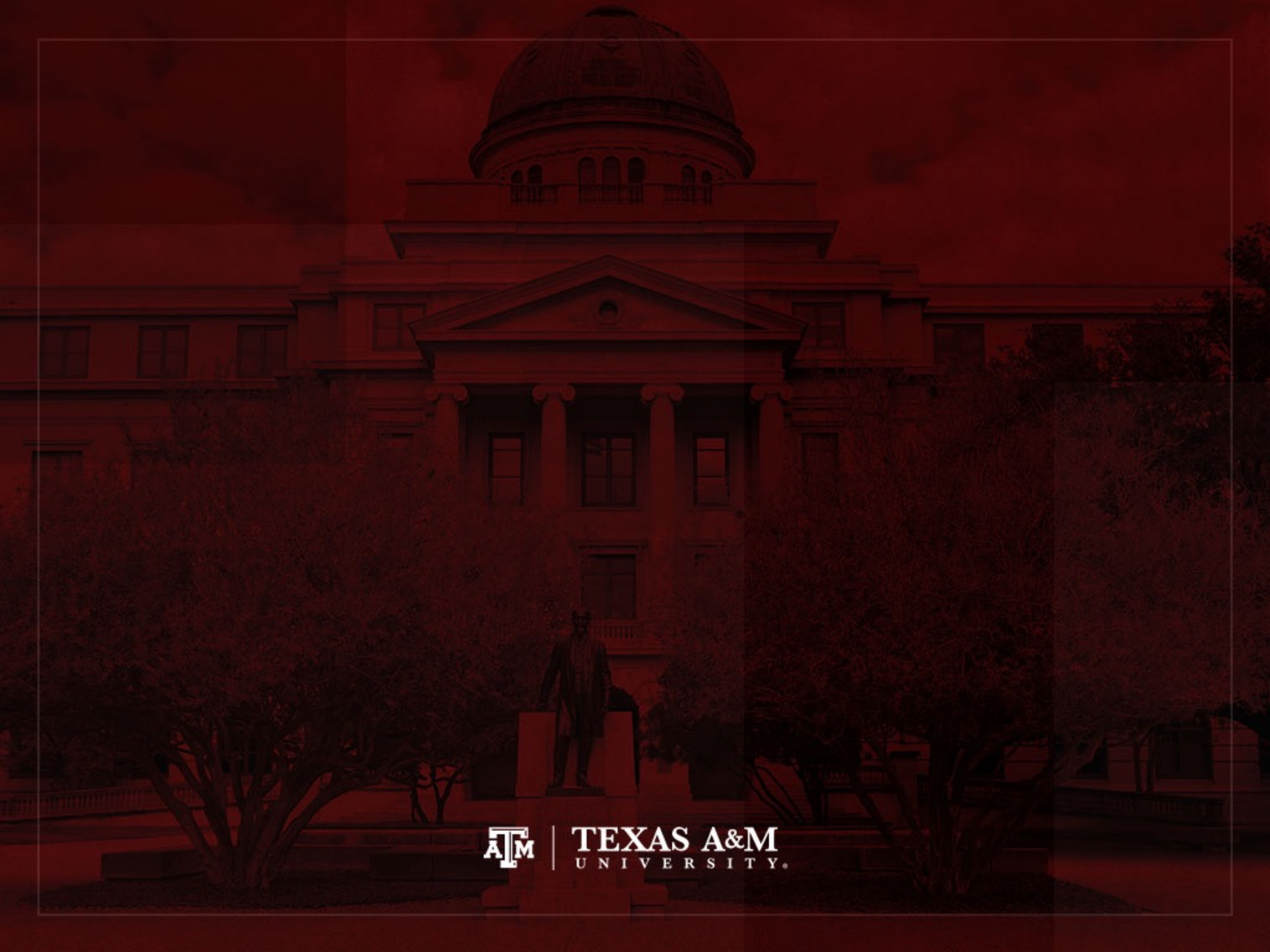
- Expectation of life at birth (e_0) is often taken as an index of overall mortality
  - However, it gives a poor idea of lifespan
  - Because it is heavily affected by infant mortality
  - *IMR* can be high in some countries

- A better index of lifespan is $10 + e_{10}$

Full cohort life table for King Edward's children

x	n	ℓ_x	${}_nq_x$	${}_nd_x$	${}_na_x$	${}_nL_x$	${}_nm_x$	T_x	e_x	$x + e_x$
0	10	10	0.100	1	0.50	90.5	0.011	334	33.4	33.4
10	10	9	0.333	3	5.33	76.0	0.039	243	27.0	37.0
20	20	6	0.167	1	10.00	110.0	0.009	167	27.8	47.8
40	20	5	0.800	4	9.00	56.0	0.071	57	11.4	51.4
60	∞	1	1.000	1	1.00	1.0	1.000	1	1.0	61.0

Source: Wachter 2014, p. 58, 60.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Shapes of lifetable functions

- Different lifetable functions express the same basic information from different points of view
- Demographers often have to
  - Start with entries for some column and work out entries for another
  - Start with bits and pieces of data from a few columns and solve for some missing piece of information
- Each lifetable function has a characteristic shape...

Typical shapes of lifetable functions

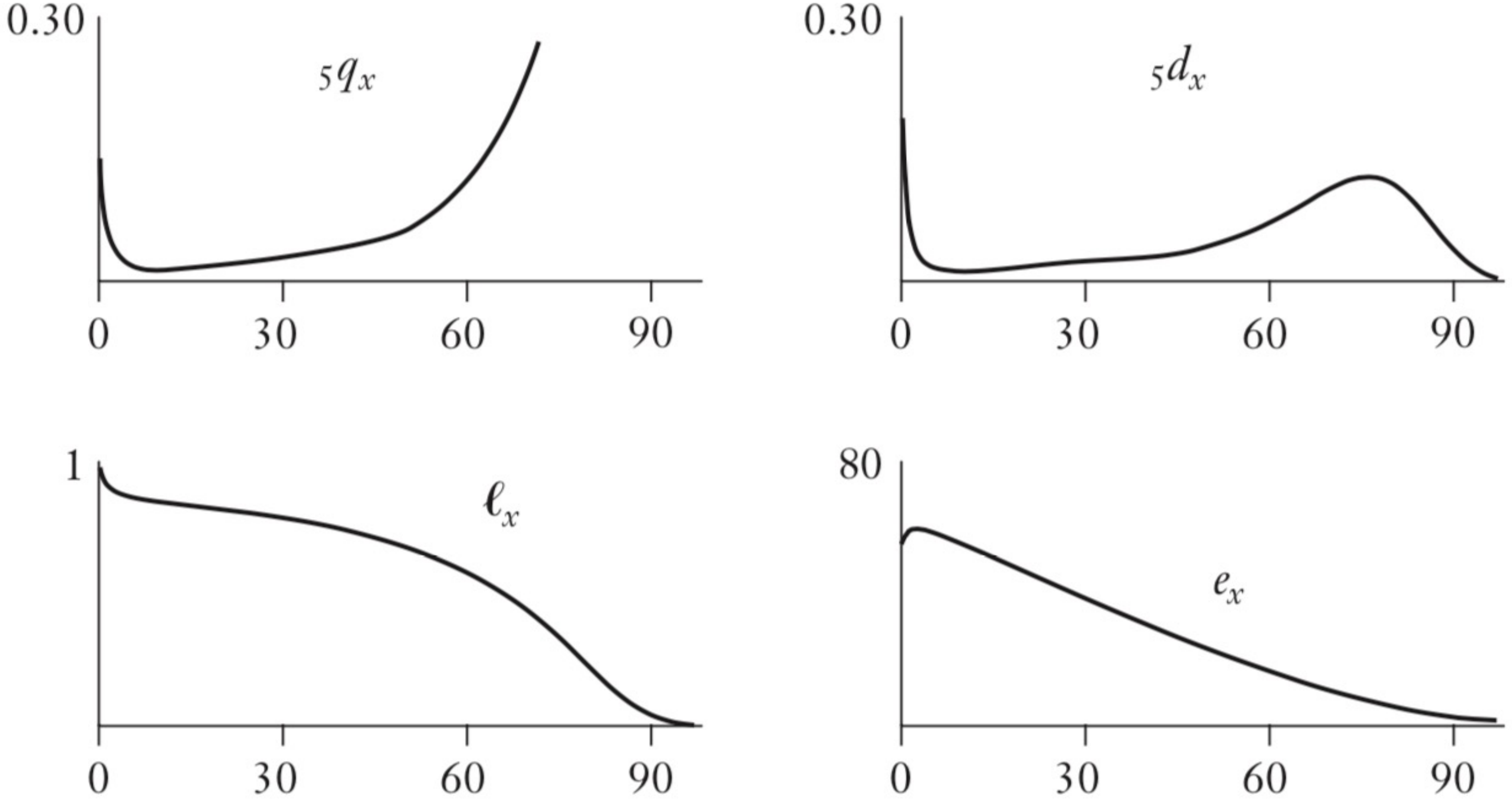


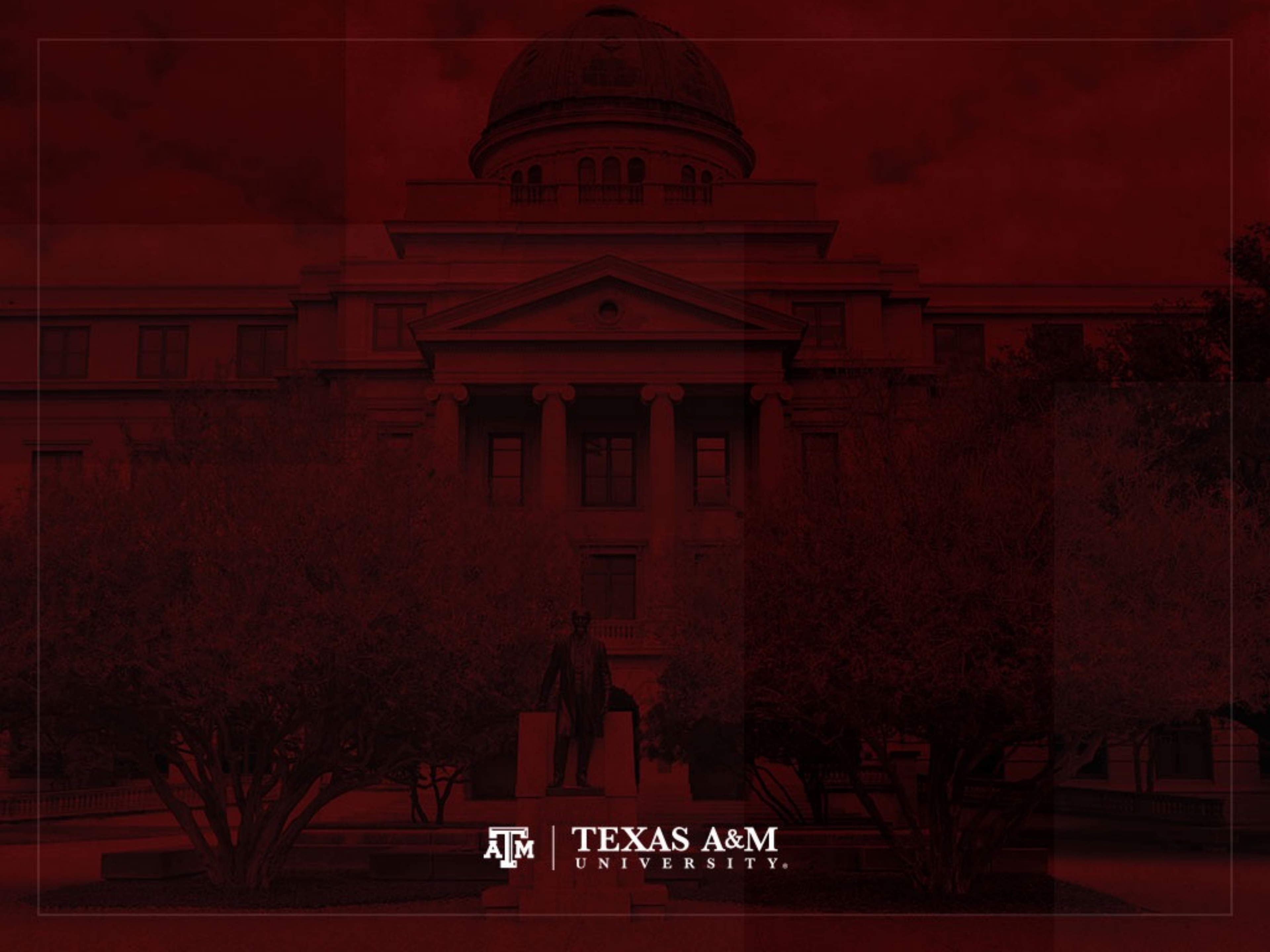
Figure 3.3 Typical shapes of lifetable functions

Source: Wachter 2014, p. 61.

Cohort lifetable formulas

Formula	Name
${}_nq_x = 1 - (\ell_{x+n}/\ell_x)$	Probability of dying
${}_nd_x = \ell_x - \ell_{x+n}$	Cohort deaths
${}_nL_x = (n)(\ell_{x+n}) + ({}_na_x)({}_nd_x)$	Person-years lived
${}_nm_x = {}_nd_x/{}_nL_x$	Lifetable death rate
$T_x = \sum_x^\infty {}_nL_a$	Remaining person-years
$e_x = T_x/\ell_x$	Expectation of life

Source: Wachter 2014, p. 62.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The radix (l_0)

- The radix (l_0) indicates the cohort's initial size
  - In Latin, it means "root"
- It does **not** have to be the size of an actual cohort
  - An initial size of 1,000 or 100,000 or 1 is easier
  - With l_0=1, l_x is the expected proportion of the cohort surviving to age x
  - Demographers choose a radix to suit their tastes

Interpreting lifetable

- The lifetable is used to follow a cohort through life
  - l_0 is seen as a random sample of the actual cohort
  - Survival of the sample mirrors survival for the whole cohort
- Conceptually, it is good to picture an actual group of people (whole cohort or sample)
  - Starting with l_0 members and living out their lives
  - Surviving
  - Aging
  - Dying

Changing the radix

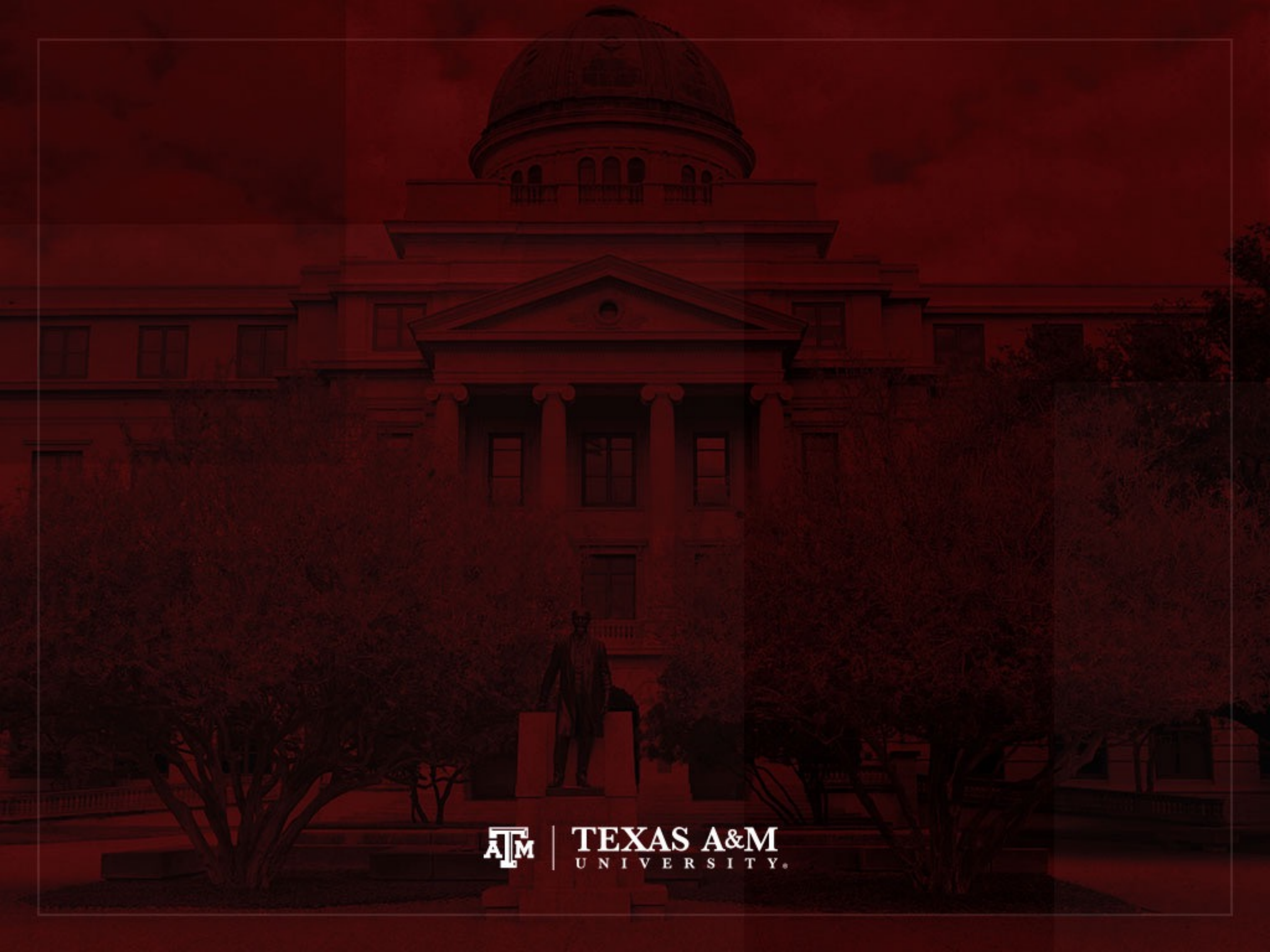
- Some quantities alter and others remain the same when changing the radix
- Quantities that change (absolute numbers)
  - l_x
  - ${}_nL_x$
  - ${}_nd_x$
- Quantities that do not change (indicators)
  - ${}_nq_x$
  - ${}_nm_x$
  - e_x

Combining lifetables

- Because men and women die at different rates, we usually construct separated lifetables by sex
  - Sometimes, we want a lifetable for everyone
  - We do not average ${}_nq_x$ or ${}_nm_x$, we work with l_x
  - Let f_{fab} be the fraction female at birth in the cohort
  - Assume that single-sex lifetables have the same radix

$$l_x^c = (f_{fab})l_x^{female} + (1 - f_{fab})l_x^{male}$$

  - l_x^c: "*c*" stands for "combined sex"
  - $(f_{fab})l_0$ baby girls
  - $(1 - f_{fab})l_0$ baby boys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Annuities and insurance

- Annuities and insurance are social institutions that become familiar usually after school or college when starting a job or a family
- Earliest applications of lifetable methods in the 1600s and 1700s were to annuities
- Idea of a steady income (e.g., after retirement)
  - You buy a policy from an annuity company for a single payment (*P*)
  - The company agrees to pay you an annual benefit (*B*) for as long as you live

Annuities

- The purpose of buying an annuity is to share risk
  - You pay now and collect benefits as long as you live
  - If you die soon, the company wins
  - If you live long, the company loses

- The company sets the purchase price to break even or come out ahead
  - Examples here deal with “actuarially fair rates”, where profit is zero
  - In practice, a margin for profit is added
  - The purchase price ought to depend on the age of the buyer

Annuities and lifetable

- To derive formulas connecting payments (*P*) and benefits (*B*), we imagine all l_x members of a cohort buying annuities at some age *x*
  - Some members live long
  - Other members don't live as long

- Over the first *n* years after purchase
  - Cohort members live a total of ${}_nL_x$ person-years
  - Each one receiving a benefit *B* each year, for $B({}_nL_x)$ in benefits overall

Formula for annuities

- Over all future ages, total benefits amount to

$$B\ {}_nL_x + B\ {}_nL_{x+n} + B\ {}_nL_{x+2n} + B\ {}_nL_{x+3n} \ldots = B\ T_x$$

- The total purchase amount equals price per person (P) times the number of buyers (l_x)

$$P(l_x)$$

- Equating purchase to benefits implies

$$P\ l_x = B\ T_x$$

$$P = B\ T_x / l_x$$

$$P = B\ e_x$$

Need to consider interests

- If an annuity with a benefit of $10,000 a year is purchased at age 20, it could cost a lot
  - If e_{20} = 50 years

$$P = B\, e_x$$

$$P = 10{,}000 * 50$$

$$P = 500{,}000$$

- Companies don't charge so much, because
  - They invest money and earn interest while it is waiting to pay future benefits
  - Time elapses between purchase and receipt of benefits

Considering interests

- To consider interest, imagine the company opening a separate bank account for each future *n*-year period
  - It invests money for early benefits in short-term investments
  - It invests money for distant future in long-term investments
- We calculate annuity price by estimating how much money the company must put into each account at the start
  - In order to have enough money to pay benefits from that account when the time comes

Interests for different accounts

- For the first account, the company has to deposit enough money to pay out benefits $B({}_nL_x)$ on average half-way through the first period
  - This leaves on average about $n/2$ years for money to grow through compound interest
  - At compound interest, 1 dollar grows to $(1 + i)^{n/2}$ dollars in $n/2$ years with interest rate i
  - So, the company needs to deposit $B\ {}_nL_x / (1 + i)^{n/2}$
- For the next account, money can earn interest for an average on $n+n/2$ years, so the deposit equals

$$B\ {}_nL_{x+n} / (1 + i)^{n+n/2}$$

- When the cohort reaches age y, the deposit is

$$B\ {}_nL_y / (1 + i)^{y-x+n/2}$$

General formula with interests

- The company needs to deposit for all accounts

$$\frac{B\ {}_nL_x}{(1+i)^{n/2}} + \frac{B\ {}_nL_{x+n}}{(1+i)^{n+n/2}} + \frac{B\ {}_nL_{x+2n}}{(1+i)^{2n+n/2}} + \cdots$$

- Lifetables with an open-ended interval starting at a top age *x*max introduce a specificity
  - The rule is to replace *n*/2 with e_{xmax}

$$(1+i)^{n+e_{xmax}}$$

  - People alive at the start of the interval will live about e_{xmax} further years

Example with King Edward's children

x	n	ℓ_x	${}_nq_x$	${}_nd_x$	${}_na_x$	${}_nL_x$	${}_nm_x$	T_x	e_x	$x+e_x$
0	10	10	0.100	1	0.50	90.5	0.011	334	33.4	33.4
10	10	9	0.333	3	5.33	76.0	0.039	243	27.0	37.0
20	20	6	0.167	1	10.00	110.0	0.009	167	27.8	47.8
40	20	5	0.800	4	9.00	56.0	0.071	57	11.4	51.4
60	∞	1	1.000	1	1.00	1.0	1.000	1	1.0	61.0

Table 3.2 Children of King Edward III of England

1330–1376	Edward, The Black Prince
1332–1382	Isabel
1335–1348	Joan
1336–?	William of Hatfield (died young)
1338–1368	Lionel of Antwerp, Duke of Clarence
1340–1398	John of Gaunt, Duke of Lancaster
1341–1402	Edmund Langley, Duke of York
1342–1342	Blanche
1344–1362	Mary
1346–1361	Margaret
1355–1397	Thomas of Woodstock, Duke of Gloucester

Source: Wachter 2014, p. 58, 60.

Example

- Suppose King Edward III had bought annuities with a benefit of £100 a year for all five of his surviving children when they were age 40
  - Interest per year = 10% = 0.1
  - ${}_{20}L_{40}$ = 56; n = 20
  - ${}_{\infty}L_{60}$ = 1; e_{xmax} = 1

$$\frac{B\ {}_nL_x}{(1+i)^{n/2}} + \frac{B\ {}_nL_{x+n}}{(1+i)^{n+e_{xmax}}} = \frac{B\ {}_{20}L_{40}}{(1+0.1)^{20/2}} + \frac{B\ {}_{\infty}L_{60}}{(1+0.1)^{20+1}} =$$

$$\frac{100\ * 56}{(1.1)^{10}} + \frac{100\ * 1}{(1.1)^{21}} = 2{,}159 + 13 = 2{,}172$$

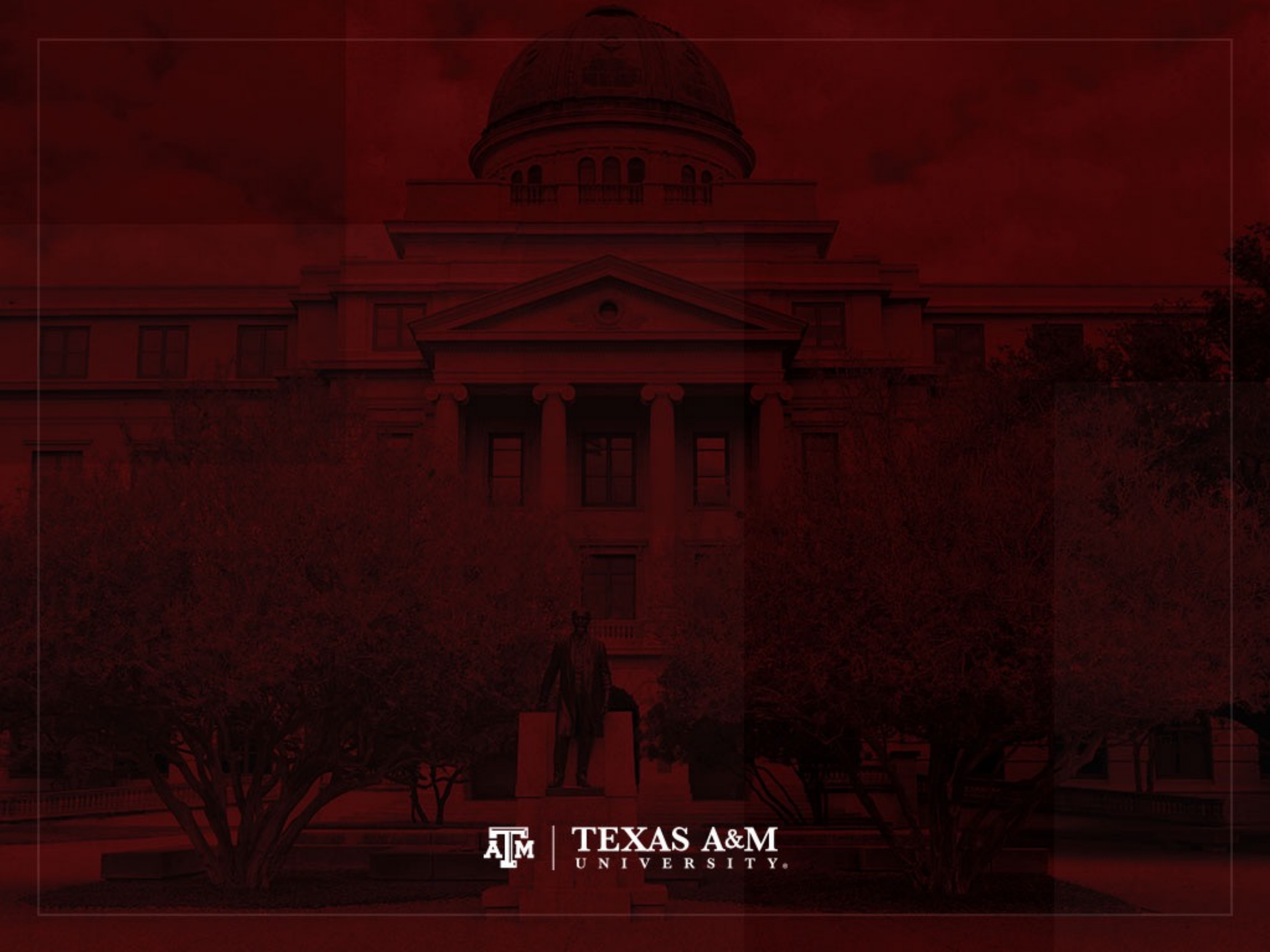
Insurance policies

- Insurance policies resemble annuities
  - But the company promises to pay an amount once when you die, not year by year when you are living
  - The purchase price (P) is paid at the start
  - All formulas are the same as for annuities
  - Except that death to cohort members (${}_nd_x$) take the place of person-years lived (${}_nL_x$)

- Today companies usually sell term insurance, where the benefit is paid only to cohort members who die in the next year (${}_1d_x$)

Variations

- Annuities may be purchased at age *x* and start paying benefits only at some later age *z*
  - This implies that the sum over terms ${}_nL_y$ only starts at $y=z$
- Buyers may have a mix of ages
  - Each age can be treated separately and results added together
- All these calculations require skills with lifetables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Mortality of the 1300s and 2000s

- The lifetable for Edward III's children is informative of mortality in England in the 1300s
  - Even with small sample of unusual people
  - Two anomalies of the data
    - Low level of infant deaths (underregistered)
    - Abbreviated life course after age 60
  - But it shows early female mortality (medieval time)

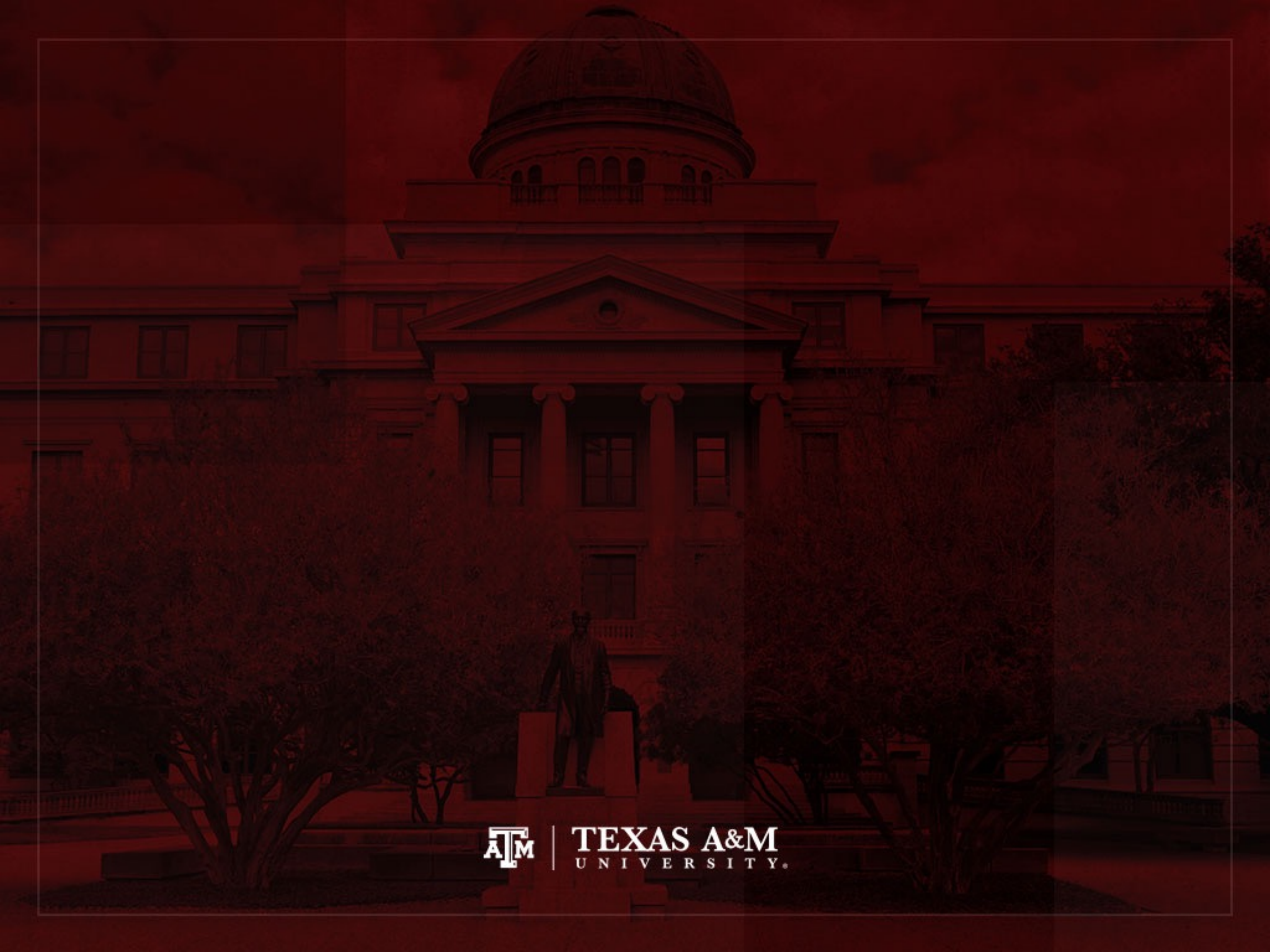
- It is interesting to think about changes between the 1300s and 2000s...

Changes in infant and old mortality

- Infant and child mortality has dropped dramatically over the last hundred years
  - Death of a baby has become an unusual event
- Life expectancies (affected by infant mortality) are no guide to maximum attained ages
  - Edward III's children lifetable: $e_0 = 33.4$
  - Edmund Langley lived past 60 (but this was rare)
- Today large numbers of people live active lives into their late 80s and 90s
  - It changes attitudes about what it means to be old

References

Wachter KW. 2014. Essential Demographic Methods. Cambridge: Harvard University Press. Chapter 3 (pp. 48–78).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