Lecture 5: Population projection

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Population projection

- Transition matrices
- Structural zeros
- The Leslie matrix subdiagonal
- The Leslie matrix first row
- Projecting fillies, mares, seniors

Transition matrices

- Transition matrices are tables used for population projection
- Official presentations of projections are often filled with disclaimers cautioning the reader that **projections** are not **predictions**
  - They do not tell us what the world **will** be like but only what the world **would** be like if a particular set of stated assumptions about future vital rates turned out to be true
- The assumptions may or may not bear any relation to what actually happens

Disingenuous disclaimers

- Projection is not just a game with computers and pieces of paper
- We do projections for a purpose to foresee the future of the population
- The choice of credible assumptions about vital rates is an important part of the art and science of projection
  - As are the formulas we use to implement the calculations
  - In this chapter, we concentrate on the formulas

Previous failures

- The record of demographers at guessing future vital rates has not been good
- The community failed to predict
  - The Baby Boom of the 1950s and 1960s
  - The Baby Lull of the 1970s and 1980s
  - It largely failed to predict the continuing trend toward lower mortality at older ages in industrialized countries
- We do not yet understand the mechanisms that drive demographic change
  - We need deeper theories with better predictive power

But we are doing good

- Despite these failures with previous predictions, demographers do better than economists, seismologists...
- Choice of assumptions about future fertility, mortality, marriage, divorce, and immigration may be difficult
- But methods for using those assumptions to calculate future population sizes and age distributions are well developed and satisfactory

Focus of this chapter

- We study these methods of calculation
- Tools based on matrices and vectors
- Projection over discrete steps of time
- Populations split up into discrete age groups

Several characteristics

- Sophisticated projections can treat a population classified by many characteristics
  - Sex, age, race, ethnicity, education, marital status, income, locality
  - So much detail is not common

- Progress is being made, under a European team led by the demographer Wolfgang Lutz
  - Incorporating education into worldwide projections

Basic ideas: simple case

- The basic ideas are illustrated by simple projections
- Focus on a single sex
- All races and ethnicities together
- We subdivide the population only by age

Leslie matrices

- The main tools for projecting the size and age distribution of a population forward through time are tables called “Leslie matrices”
  - By P.H. Leslie (1945)
  - Same approach done few years earlier by H. Bernardelli and E.G. Lewis

- Related to Markov chains in probability theory
  - But what is being projected with Leslie matrices are expected numbers of individuals rather than probabilities

Leslie matrices for age structure

- Leslie matrices are a special case of transition matrices

- Demographers use general transition matrices
  - To project a distribution of marital status, parity, education, or other variables into the future

- They use the special transition matrices that Leslie defined
  - To project age structure

Defining a transition matrix

- A transition matrix is a table with rows and columns showing the expected number of individuals
  - who **end** up in the state with the label on the **row**
  - per individual at the **start** in the state with the label on the **column**

$$\begin{array}{c|c} & \begin{array}{ccccc} \text{0 to 5} & \text{5 to 10} & \text{10 to 15} & \text{15 to 20} & \text{20 to 25} \end{array} \\ \begin{array}{c} \text{0 to 5} \\ \text{5 to 10} \\ \text{10 to 15} \\ \text{15 to 20} \\ \text{20 to 25} \end{array} & \begin{pmatrix} \text{kids} & \text{kids} & \text{kids} & \text{kids} & \text{kids} \\ \text{survivors} & 0 & 0 & 0 & 0 \\ 0 & \text{survivors} & 0 & 0 & 0 \\ 0 & 0 & \text{survivors} & 0 & 0 \\ 0 & 0 & 0 & \text{survivors} & 0 \end{pmatrix} \end{array}$$

One-step transition

- A Leslie matrix is a special case of a transition matrix in which the states correspond to age groups
  - Processes of transition are surviving and giving birth

- The Leslie matrix describes a one-step transition
  - We project the population forward one step at a time
  - The time between **start** and **end** (projection step) should be equal to the **width** (*n*) of all age groups

Age group width

- The fact that the step size has to equal the age group width is crucial
  - Generally pick one sex, usually females
  - Divide the female population into age groups of width *n*
  - Width may be 1, 5, 15 years, or some other number

Examples

- If we are using 1-year age groups
  - We have to project forward 1 year at a time
  - To project 10 years into the future requires 10 projection steps

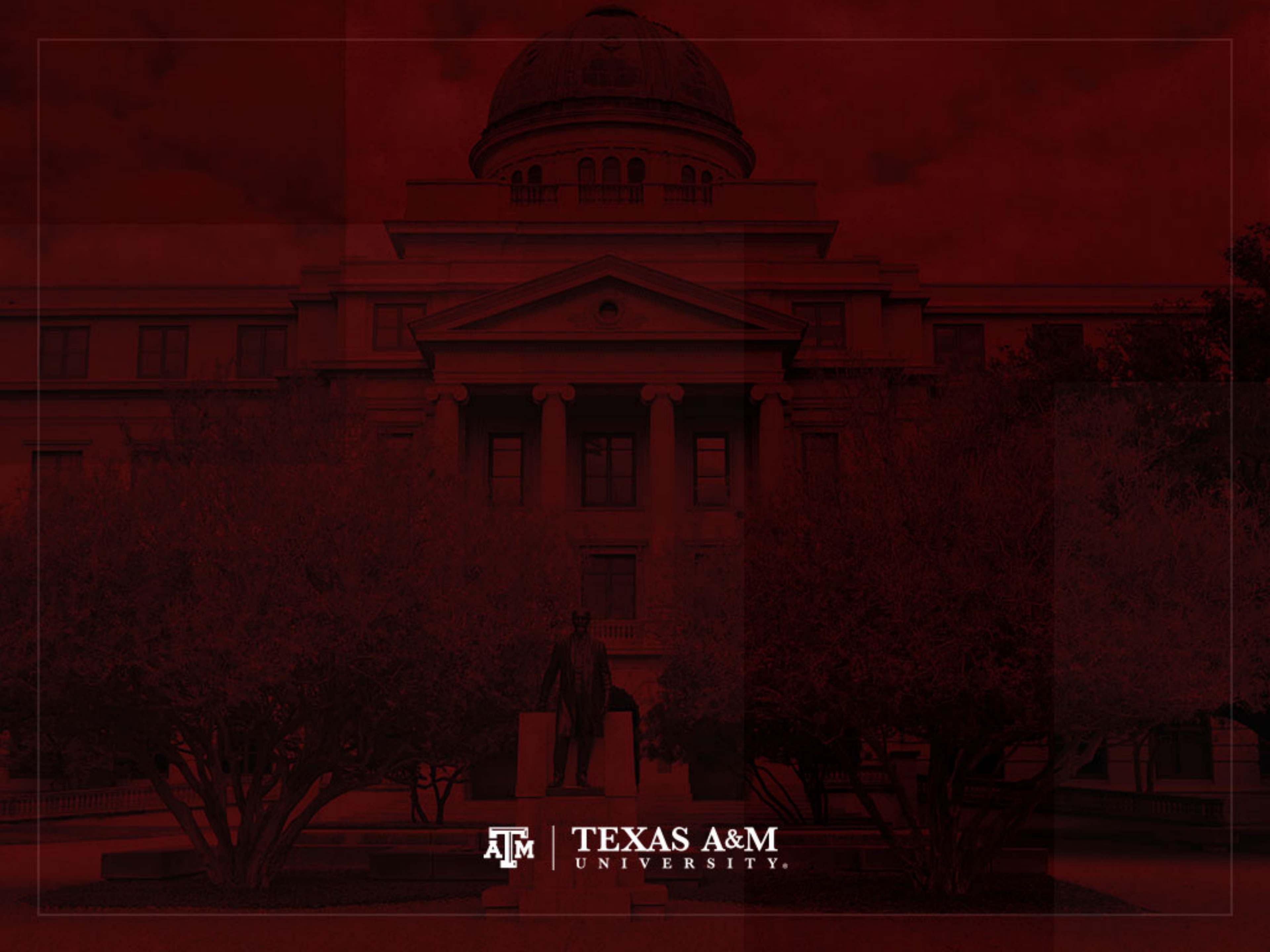
- If we are using 5-year age groups
  - We have to project forward 5 years at a time
  - To project 10 years into the future requires only 2 projection steps

Closed population

- We assume a closed population
- No migrants are included in projections
- People enter the population only by being born to members already in the population
- People leave it only by dying

Childbirth as a transition

- Projections treat childbirth as a possible transition along with survival
  - People who end up in some state may not be the same people who start in any one of the states
  - They may be the babies of people who start in the various states
- We are concerned with the expected numbers
  - In the state for the row (end)
  - Per person in the state for the column (start)
  - Without regard to how the people are channeled there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Structural zeros

- The logic of the transition process is built into a transition matrix through the pattern of zeros
- Some age groups owe no part of their numbers at the end of the step to certain other age groups
- Suppose we have n=5
  - We have 5-year-wide age groups
  - We are projecting forward 5 years in one step

Example of teenagers

- No teenagers owe their numbers to 40-year-olds five years before
- The value of the Leslie matrix element has to be zero in
  - The row for 15 to 20-year-olds (end)
  - The column for 40-year-olds (start)
- This is a “structural zero”
  - We know because of the logic of the processes of aging and childbirth

Example of a Leslie matrix

- Most of the elements of a Leslie matrix are structural zeros
  - We can fill them in immediately
  - At the end of 5 years, people age up to 5 years
  - No one can get younger over time

Start

End

$$
\begin{array}{c|ccccc}
 & \text{0 to 5} & \text{5 to 10} & \text{10 to 15} & \text{15 to 20} & \text{20 to 25} \\
\hline
\text{0 to 5} & \text{kids} & \text{kids} & \text{kids} & \text{kids} & \text{kids} \\
\text{5 to 10} & \text{survivors} & 0 & 0 & 0 & 0 \\
\text{10 to 15} & 0 & \text{survivors} & 0 & 0 & 0 \\
\text{15 to 20} & 0 & 0 & \text{survivors} & 0 & 0 \\
\text{20 to 25} & 0 & 0 & 0 & \text{survivors} & 0
\end{array}
$$

Following the same logic

- Below the first row, all elements are structural zeros except the subdiagonal
- No one can
  - Jump an age group
  - Stay in the same age group
  - Get younger
- They can only move into the next age group
  - If they survive
  - It is important to have the same age-group width

First row

- What about the first row, for people who end up aged 0 to 5 at the end of 5 years?
  - No one can survive into this row
  - These elements are not structural zeros
  - There can be babies born during the projection step who are found in this age group at the end of the step
  - The number of babies depends on the number of potential parents in the various age groups at the start

First row of a Leslie matrix

Potential parents at the start

Babies at the end

$$\begin{array}{cc}
 & \begin{array}{ccccc} \text{0 to 5} & \text{5 to 10} & \text{10 to 15} & \text{15 to 20} & \text{20 to 25} \end{array} \\
\begin{array}{c} \text{0 to 5} \\ \text{5 to 10} \\ \text{10 to 15} \\ \text{15 to 20} \\ \text{20 to 25} \end{array} &
\begin{pmatrix}
\text{kids} & \text{kids} & \text{kids} & \text{kids} & \text{kids} \\
\text{survivors} & 0 & 0 & 0 & 0 \\
0 & \text{survivors} & 0 & 0 & 0 \\
0 & 0 & \text{survivors} & 0 & 0 \\
0 & 0 & 0 & \text{survivors} & 0
\end{pmatrix}
\end{array}$$

Upper-left element

- The upper-left element equals zero depending on the age-group width
  - If n=5, we do not expect there to be any babies in 5 years to people 0 to 5 at the start
  - If n=15, we do expect babies in the next 15 years to people aged 0 to 15 at the start
- This element also depends on empirical knowledge about youngest ages of childbearing
  - It is often equal to zero
  - But it is not regarded as a structural zero

Representing structural zeros

- Another way of representing information about structural zeros is a diagram of permitted transitions

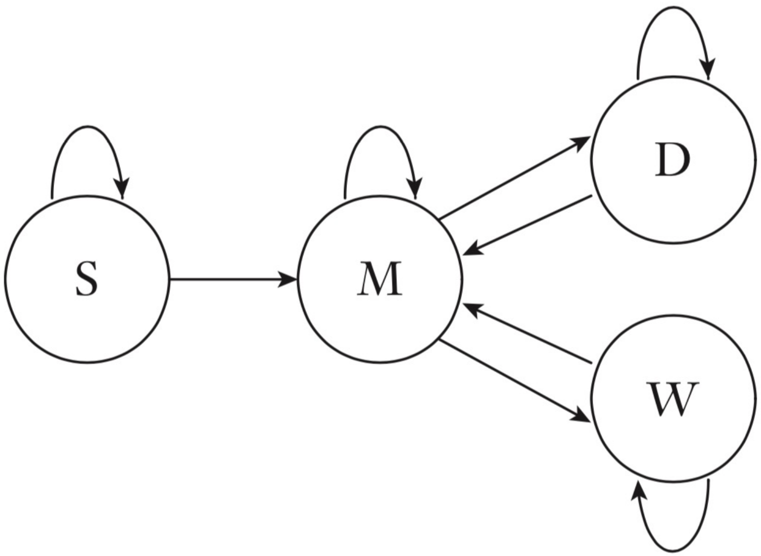
- We mark states within circles and draw an arrow from one state to another if there is a nonzero element for that column-row pair
  - Show links from individuals in the sender state at the beginning of the arrow
  - To individuals who can show up in the receiver state at the end of the arrow

Arrow diagrams

- Arrow diagrams are helpful when transitions are not between age groups
- Useful for transitions between states with a logic of their own
- Same idea as programmer flow charts

Example for marital status

- Four states
  - Single (S), never married
  - Married (M)
  - Widowed (W)
  - Divorced (D)

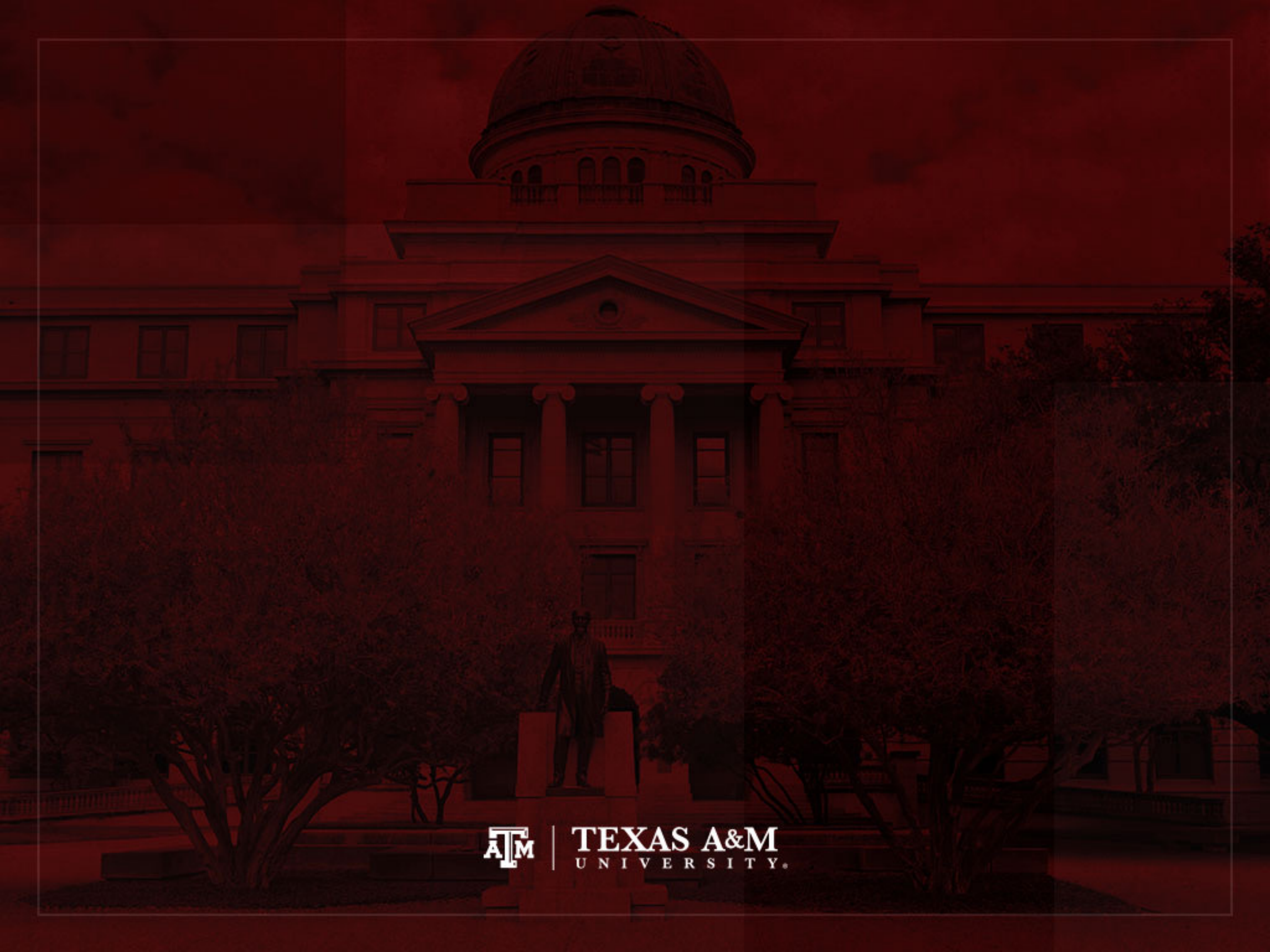


- Suppose the projection step is too short
  - Nobody can get both married and divorced, or both divorced and remarried within a single step
  - Multiple transitions within one step are not numerically significant

Marital status transition matrix

- The structural zeros in the transition matrix corresponding to the previous diagram appear in slots marked “0”

$$
\begin{array}{c|cccc}
 & \text{Single} & \text{Married} & \text{Widowed} & \text{Divorced} \\
\hline
\text{Single} & x & 0 & 0 & 0 \\
\text{Married} & x & x & x & x \\
\text{Widowed} & 0 & x & x & 0 \\
\text{Divorced} & 0 & x & 0 & x
\end{array}
$$



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The Leslie matrix subdiagonal

- We generally denote a matrix by a single capital letter like A
  - The first subscript is for the row and the second subscript is for the column
  - $A_{3\,2}$ element in third row and second column, survivors from the second age group to the third age group
- This notation for matrix elements is universal
  - $A_{\text{to, from}}$ or $A_{\text{row, column}}$
  - Subscript for the destination age group comes first
  - Subscript for the origin age group comes second

Developing a formula

- Develop a formula for the elements along the subdiagonal of the Leslie matrix
  - They represent transitions of survival
  - Continue to use 5-year-wide age groups

Start

$$
\textbf{End}\quad
\begin{array}{c c}
 & \begin{array}{ccccc} \text{0 to 5} & \text{5 to 10} & \text{10 to 15} & \text{15 to 20} & \text{20 to 25} \end{array} \\
\begin{array}{c} \text{0 to 5} \\ \text{5 to 10} \\ \text{10 to 15} \\ \text{15 to 20} \\ \text{20 to 25} \end{array} &
\begin{pmatrix}
\text{kids} & \text{kids} & \text{kids} & \text{kids} & \text{kids} \\
\text{survivors} & 0 & 0 & 0 & 0 \\
0 & \text{survivors} & 0 & 0 & 0 \\
0 & 0 & \text{survivors} & 0 & 0 \\
0 & 0 & 0 & \text{survivors} & 0
\end{pmatrix}
\end{array}
$$

Example

- Consider $A_{3\ 2}$, the expected number of people
  - Aged 10 to 15 at the end (row)
  - Per person aged 5 to 10 at the start (column)
- Age group of 5-to-10-year-olds at the start, at time t (“today”)
  - Composed of cohorts born between times $t-10$ and $t-5$
- We follow the experience of five 1-year birth cohorts on the Lexis diagram...

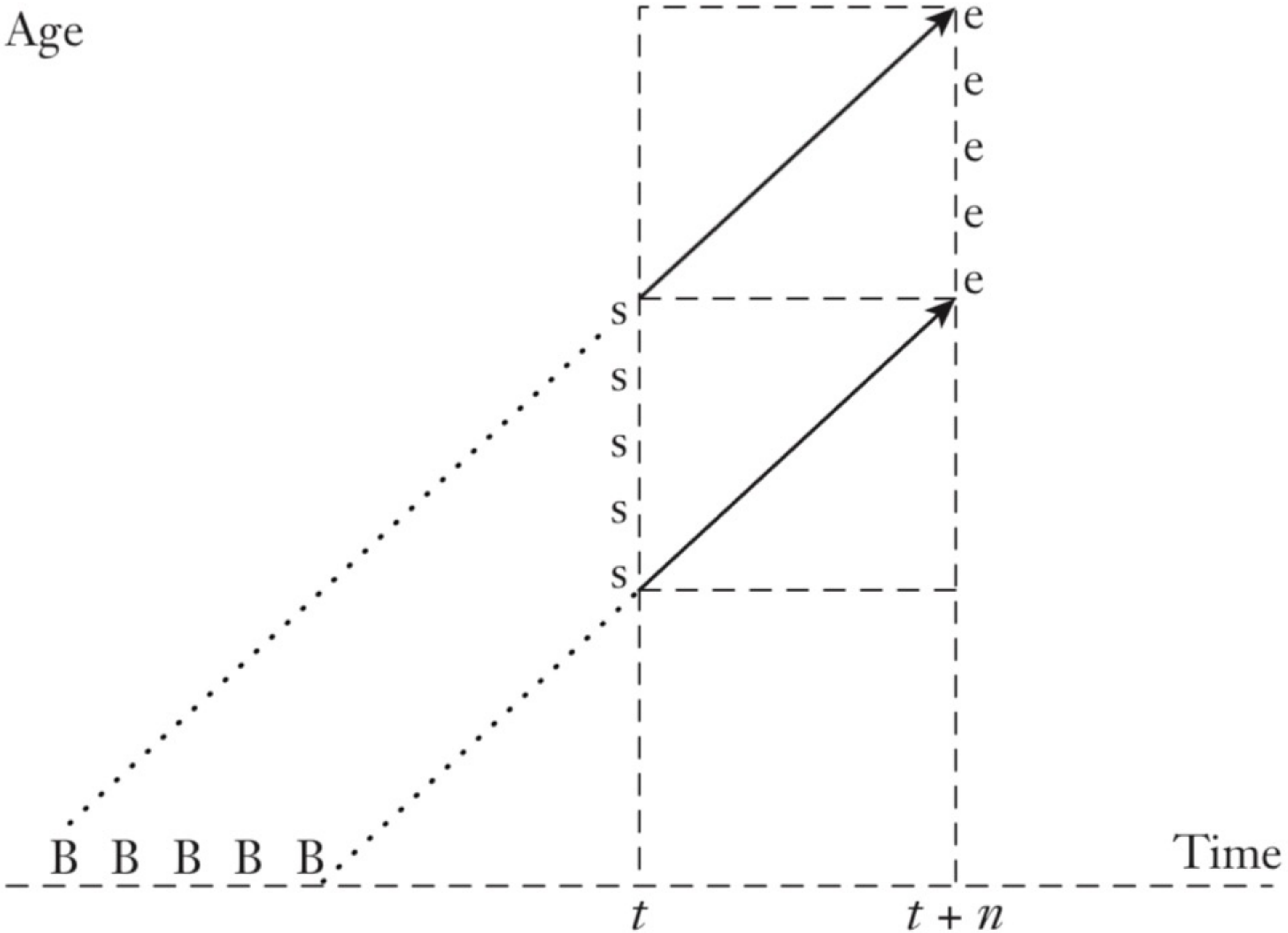


Figure 5.2 Contributions to the Leslie matrix subdiagonal

Symbols

- “B” symbols represent single-year cohorts at birth
- “s” symbols represent cohorts at the start of the projection step
  - 5–10 age group
- “e” symbols represent cohorts at the end of the projection step
  - 10–15 age group

Assumptions

- We ignore changes in sizes of cohorts at birth inside each 5-year period
  - We pretend that all changes in initial cohort sizes occur in jumps between periods

- We also assume that the same lifetable (mortality)
  - Applies to all the 1-year cohorts in this 5-year group of cohorts
    - From 5–10 age group to 10–15 age group

Cohorts in the diagram

- Examples of some of the cohorts on the diagram
- The earliest/oldest cohort (now 10-year-olds)
  - They had some size l_0 at birth
  - At the start of the projection step (s), l_{10} members left
  - At the end of the projection step (e), l_{15} members left
- The latest/youngest cohort (now 5-year-olds)
  - They had the same size l_0 at birth
  - Now (s), l_5 members left
  - Five years from now (e), l_{10} members left

Size of 5-year age group

- The whole 5-year age group is about five times as large as the average size of its youngest and oldest cohorts

- At the start of the projection step (*s*): *t*

$$(5/2)\ (l_5 + l_{10})$$

- At the end of the projection step (*e*): *t* + *n*

$$(5/2)\ (l_{10} + l_{15})$$

Person-years lived

- The sizes of the 5-year age group at the start and end are approximations for person-years lived

$$ {}_5L_5 = (5/2)\,(l_5 + l_{10}) $$

$$ {}_5L_{10} = (5/2)\,(l_{10} + l_{15}) $$

- If the age group was split into many small cohorts
  - We could add them all up
  - We would have obtained the ${}_nL_x$ values exactly

Subdiagonal as ratios

- The subdiagonal element $A_{3\,2}$ of the Leslie matrix
  - Ratio of the 10–15-year-olds at the end
  - To the 5–10-year-olds at the start

$$A_{3\,2} = \frac{{}_5L_{10}}{{}_5L_5}$$

$$\begin{array}{c} \\ \text{0 to 5} \\ \text{5 to 10} \\ \text{10 to 15} \\ \text{15 to 20} \\ \text{20 to 25} \end{array}
\begin{array}{c}
\begin{array}{ccccc} \text{0 to 5} & \text{5 to 10} & \text{10 to 15} & \text{15 to 20} & \text{20 to 25} \end{array} \\
\begin{pmatrix}
\text{kids} & \text{kids} & \text{kids} & \text{kids} & \text{kids} \\
\frac{{}_5L_5}{{}_5L_0} & 0 & 0 & 0 & 0 \\
0 & \frac{{}_5L_{10}}{{}_5L_5} & 0 & 0 & 0 \\
0 & 0 & \frac{{}_5L_{15}}{{}_5L_{10}} & 0 & 0 \\
0 & 0 & 0 & \frac{{}_5L_{20}}{{}_5L_{15}} & 0
\end{pmatrix}
\end{array}$$

Ratio of *L* values

- Each subdiagonal element of a Leslie matrix is a ratio of big-*L* values
  - The age label on the **numerator** comes from the **row**
  - The age label on the **denominator** comes from the **column**

$$
\begin{array}{c|c}
 & \begin{array}{ccccc} 0 \text{ to } 5 & 5 \text{ to } 10 & 10 \text{ to } 15 & 15 \text{ to } 20 & 20 \text{ to } 25 \end{array} \\
\begin{array}{c} 0 \text{ to } 5 \\ 5 \text{ to } 10 \\ 10 \text{ to } 15 \\ 15 \text{ to } 20 \\ 20 \text{ to } 25 \end{array} &
\begin{pmatrix}
\text{kids} & \text{kids} & \text{kids} & \text{kids} & \text{kids} \\
\frac{{}_5L_5}{{}_5L_0} & 0 & 0 & 0 & 0 \\
0 & \frac{{}_5L_{10}}{{}_5L_5} & 0 & 0 & 0 \\
0 & 0 & \frac{{}_5L_{15}}{{}_5L_{10}} & 0 & 0 \\
0 & 0 & 0 & \frac{{}_5L_{20}}{{}_5L_{15}} & 0
\end{pmatrix}
\end{array}
$$

General notation

- The bottom age (denominator) of the age group x is expressed as $x = jn - n$
  - In terms of the column number j

$$A_{j+1,j} = \frac{{}_nL_{x+n}}{{}_nL_x}$$

- $A_{3\,2}$: from 5–10 (denominator) to 10–15 (numerator)
  - $x = jn - n = 2(5) - 5 = 10 - 5 = 5$

$$A_{3\,2} = \frac{{}_5L_{10}}{{}_5L_5}$$

Different cohorts

- The cohort born between $t–x–n$ and $t–x$ supplies the numerator and denominator in this element

$${}_nL_{x+n} \,/\, {}_nL_x$$

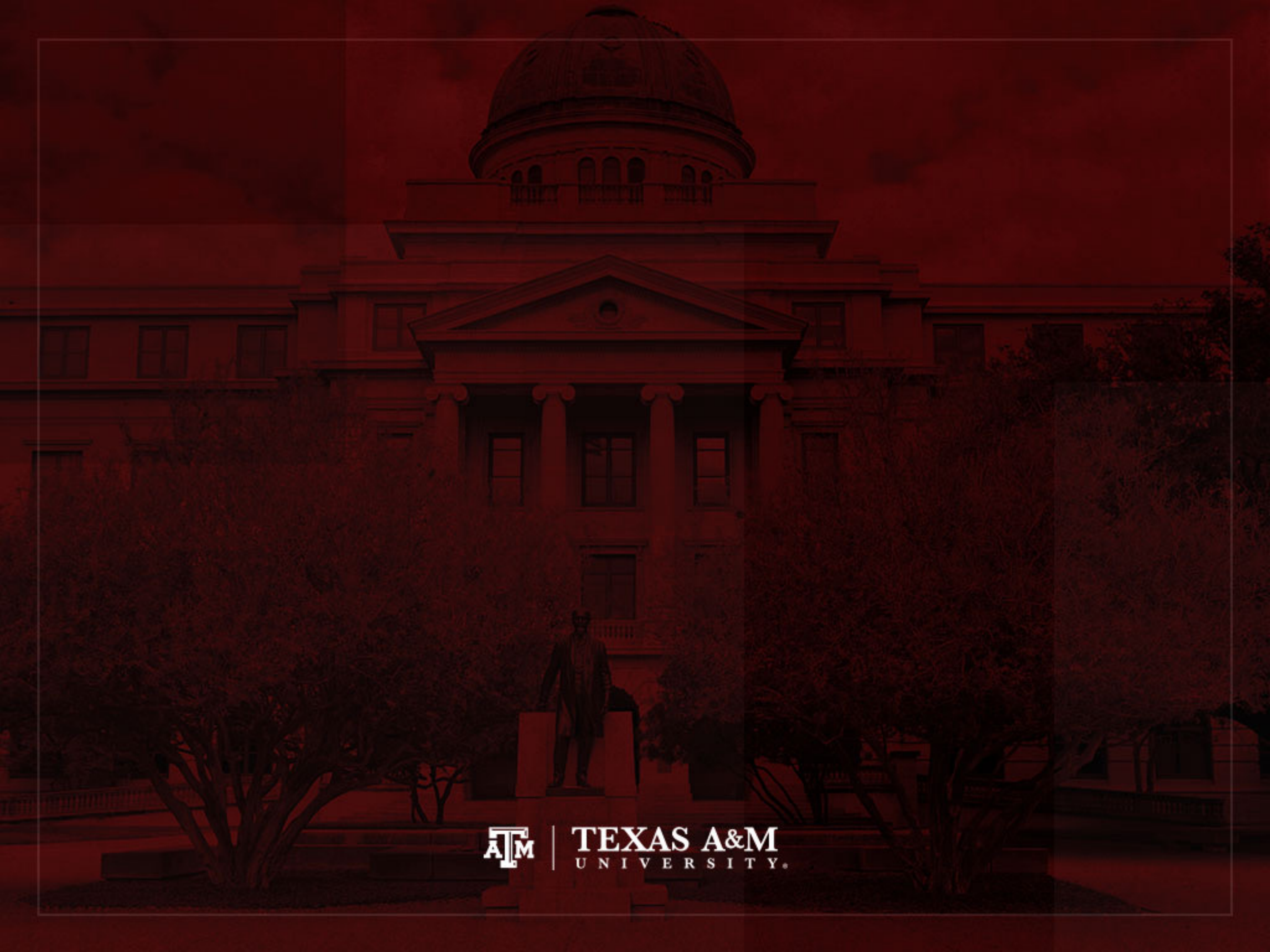
- There is a different cohort for each age group x
  - L values for different columns of the Leslie matrix are L values from different cohort lifetables
  - They are L values from a period lifetable (chapter 7)

Period lifetable

- Period lifetable puts together survival experience for different cohorts as they move through the same time period

- The Leslie matrix does the same

- At this stage, the key concept is the way that ratios of big-*L* values supply elements for the subdiagonal matrix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The Leslie matrix first row

- Outside the subdiagonal, the only elements in a Leslie matrix which are not structural zeros are in the first row
- These are the elements that take account of population renewal
  - Babies are born to parents
  - They survive and counted at the end of projection step
- Formulas for the first row are more complicated than those for the subdiagonal
  - Let’s develop them one step at a time

Projecting female population

- Formulas are shown to project female population
  - Daughters are in first row, instead of sons and daughters
  - In projections, we must have the same kinds of people coming out as going in

- Joint projections for males and females are possible in principle
  - Simple and appealing two-sex model is a problem that remains largely unsolved

Projecting male population

- One-sex projections could be done for males
  - Inserting fertility rates for sons and fathers
  - But motherhood ages are more regular than fatherhood ages
- Usually the female population is projected
  - Counts of males are estimated from projected counts for females

Fraction female at birth

- When projecting females, we must remember that we need fertility rates for daughters only
- Published fertility rates are usually for babies of both sexes
- Need to multiply by the fraction female at birth (f_{fab})
  - By our default, it is the fraction 0.4886

Full formula for first row

- Formula for the expected number of daughters aged 0 to *n* at the end of the projection step
  - Per woman aged *x* to *x*+*n* at the start

- We write *j*(*x*) for the corresponding column with *x*=*j*(*x*)(*n*)−*n*

$$A_{1,\,j(x)} = \frac{{}_nL_0}{2\,\ell_0}\left({}_nF_x + {}_nF_{x+n}\frac{{}_nL_{x+n}}{{}_nL_x}\right) f_{\mathrm{fab}}$$

Formula step by step

- $A_{1,\, j(x)}$ pertains to
  - Daughters entering the population over n years
  - By being born to mothers aged x to x+n at the start

Potential mothers at the start (x to x+n)

Daughters at the end (0 to n)

$$\begin{array}{c} \\ \text{0 to 5} \\ \text{5 to 10} \\ \text{10 to 15} \\ \text{15 to 20} \\ \text{20 to 25} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{0 to 5} & \text{5 to 10} & \text{10 to 15} & \text{15 to 20} & \text{20 to 25} \end{array} \\ \begin{pmatrix} \text{kids} & \text{kids} & \text{kids} & \text{kids} & \text{kids} \\ \text{survivors} & 0 & 0 & 0 & 0 \\ 0 & \text{survivors} & 0 & 0 & 0 \\ 0 & 0 & \text{survivors} & 0 & 0 \\ 0 & 0 & 0 & \text{survivors} & 0 \end{pmatrix} \end{array}$$

Crude version of formula

- How many daughters per potential mother should there be?
- A first guess would be to multiply
  - The daughters-only age-specific fertility rate $({}_nF_x)(f_{\text{fab}})$ for women x to $x+n$
  - By the years at risk in the interval (n)
- ${}_nF_x$ indicates period age-specific fertility rate
  - Instead of small ${}_nf_x$ for cohort age-specific fertility rate
- It provides a first crude version of the formula

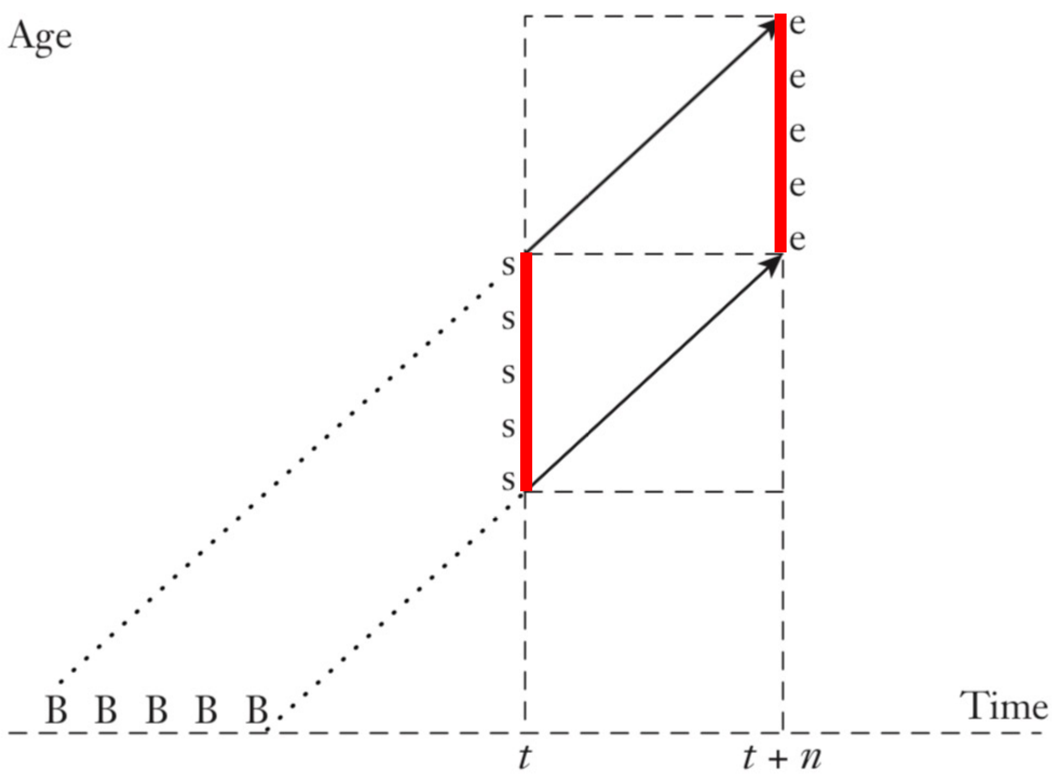
$$(n)({}_nF_x)(f_{\text{fab}})$$

Consider mortality of daughters

- Need to recognize that not all baby daughters survive to the end of the projection interval
  - The first age group counts kids aged 0 to *n* at the end of the step, not newborns
  - Need to estimate proportion of babies born during the *n* years who survive to be counted
  - Babies born early in the period have to survive to be nearly *n* years old
  - Babies born late in the period have to survive only a little while

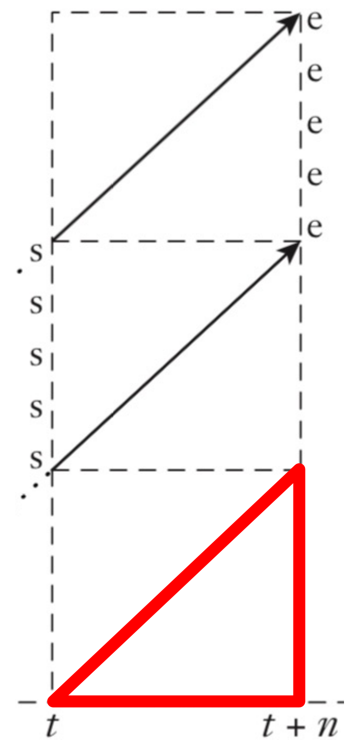
Survivorship for subdiagonal

- We averaged survivorships when we were finding a formula for subdiagonal elements of mothers
  - Compare lifelines crossing **two sides** of parallelogram



Survivorship for newborns

- We need average of survivorships for <u>daughters</u>
  - They start at birth, bottom axis of the Lexis diagram
  - Compare lifelines crossing two adjacent sides of a **right triangle**
  - Triangle base covers the interval from *t* to *t*+*n*
  - Perpendicular side reaches up from age 0 to age *n* above time *t*+*n*



Average of survivor daughters

- We ignore changes in initial cohort size within the interval
  - Out of any l_0 girls born near the start of the interval, about l_n survive to the end
  - Out of any l_0 born close to the end, nearly all l_0 survive to the end
- From $(n)(l_0)$ births, we expect this average of daughters who survive
  - $(n)(l_n + l_0) / 2$
  - This is a standard approximation for ${}_nL_0$

Better version of formula

- Average of daughters who survive: ${}_nL_0 = (n)(l_n+l_0)/2$
- Total births: $(n)(l_0)$
- Ratio of survivors to births: ${}_nL_0 \,/\, (n)(l_0)$
- Multiply this ratio by the first crude formula

$$\left(\frac{{}_nL_0}{(n)(\ell_0)}\right)(n)({}_nF_x)(f_{\text{fab}})$$

Mortality of daughters

Crude formula

n terms will cancel out

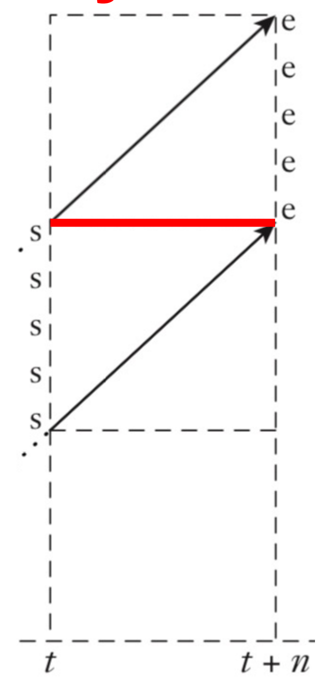
Consider aging of mothers

- Consider the aging of potential mothers during the projection interval
  - Women aged x to $x+n$ at the start only spend on average about half of the next n years in their starting age group
  - They grow older and spend about half the interval in the next age group

- In place of n years at ${}_nF_x$: $(n)({}_nF_x)$
  - We have about $n/2$ years at ${}_nF_x$: $(n/2)({}_nF_x)$
  - We have about $n/2$ years at ${}_nF_{x+n}$: $(n/2)({}_nF_{x+n})$

Consider mortality of mothers

- *n*/2 and n/2 are not quite the right breakdown
  - Not all women survive into the next age group
- We pretend that all deaths between start (s) and end (e) happen at the **age-group boundary** at age $x+n$
  - Then fraction of women surviving into the next age group is ${}_nL_{x+n}$ divided by ${}_nL_x$
  - It reduces time spent in the older age group (${}_nF_{x+n}$) by this survival fraction (${}_nL_{x+n}$ / ${}_nL_x$)



Aging and mortality of mothers

- Consider aging of mothers
  - Years spent in starting and ending age groups (*n*/2)
- Consider mortality of mothers
  - Reduce time spent in the older age group (${}_nF_{x+n}$) by this survival fraction (${}_nL_{x+n}$ / ${}_nL_x$)

All elements of formula

Women live half of the time
in starting and next age groups

$$\boxed{\left(\frac{{}_nL_0}{(n)(\ell_0)}\right)}\left(\boxed{{}_nF_x + {}_nF_{x+n}}\boxed{\frac{{}_nL_{x+n}}{{}_nL_x}}\right)\boxed{f_{\text{fab}}}\boxed{\left(\frac{n}{2}\right)}$$

Mortality
of daughters

Aging
of mothers

Mortality
of mothers

Only
daughters

All elements of formula

Women live half of the time
in starting and next age groups

$$\left(\frac{{}_nL_0}{(\cancel{n})(\ell_0)}\right)\left({}_nF_x + {}_nF_{x+n}\frac{{}_nL_{x+n}}{{}_nL_x}\right) f_{\text{fab}} \left(\frac{\cancel{n}}{2}\right)$$

Mortality
of daughters

Aging
of mothers

Mortality
of mothers

Only
daughters

All elements of formula

Women live half of the time
in starting and next age groups

$$\underbrace{\left(\frac{{}_nL_0}{(\cancel{n})(\ell_0)}\right)}_{\text{Mortality of daughters}}\left(\underbrace{{}_nF_x + {}_nF_{x+n}}_{\text{Aging of mothers}}\underbrace{\frac{{}_nL_{x+n}}{{}_nL_x}}_{\text{Mortality of mothers}}\right)\underbrace{f_{\text{fab}}}_{\text{Only daughters}}\left(\frac{\cancel{n}}{2}\right)$$

Mortality
of daughters

Aging
of mothers

Mortality
of mothers

Only
daughters

$$A_{1,j(x)} = \frac{{}_nL_0}{2\,\ell_0}\left({}_nF_x + {}_nF_{x+n}\frac{{}_nL_{x+n}}{{}_nL_x}\right) f_{\text{fab}}$$

Fertility of youngest age group

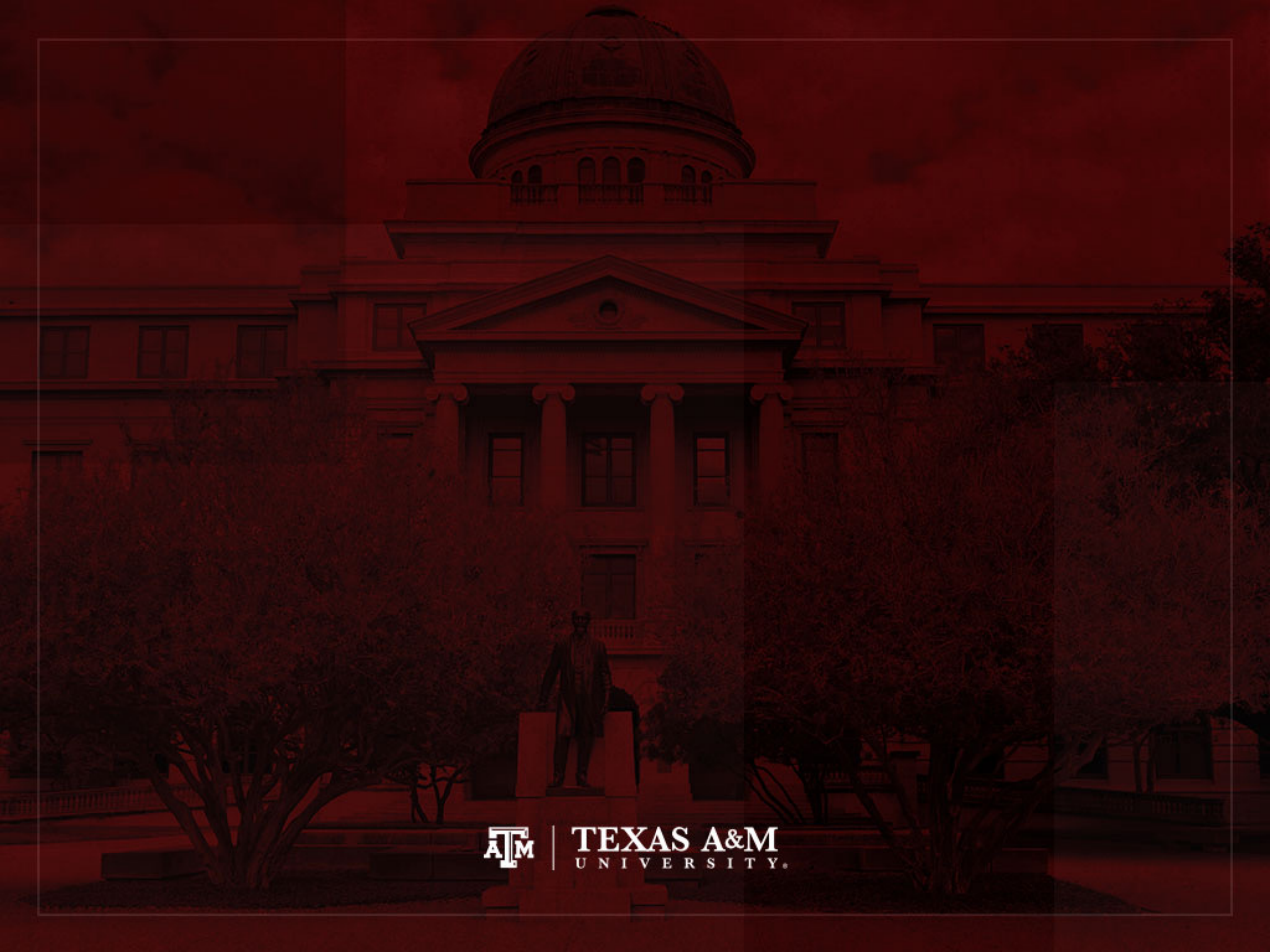
- Usually the interval width (*n*) is 1 or 5 years
  - Age-specific fertility for the youngest age group from 0 to *n* will be zero

- The youngest age group is part of the "preprocreative span"
  - Period prior to procreation (production of infants)

- With wider intervals (10, 15, 20...), fertility for the youngest age group will not equal zero

Correction for youngest group

- With wider intervals (10, 15, 20...)
  - Children born during the projection step could grow up and bear their own children during one projection step
- Formulas made no allowance for grandchildren
  - A rough correction is to insert (2)(${}_{n}F_{0}$), instead of ${}_{n}F_{0}$ for the upper-left element of the Leslie matrix
  - When fertility before age n is zero, this change has no effect
  - When it is not zero, the correction improves accuracy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Projecting fillies, mares, seniors

- Example of a Leslie matrix and a population projection employing this matrix
  - No more than three rows and columns
  - Retain familiar 5-year age group
- Population of horses in a stable, with rates of survival and fertility that are stylized but credible
  - Fillies: young female horses (0–5)
  - Mares: mature females (5–10)
  - Seniors (10–15)

Age of reproduction

- Horses mature fairly quickly and may live to ages like 30
- In our stable they only give birth to offspring (foals) between ages 5–15
- We only project the population up to age 15
  - Leslie matrices often stop at the last age of reproduction
  - Sizes of older age groups can be computed directly from the relevant lifetable

Data for this example

- Age-specific fertility rates are

$${}_5F_0\ f_{\text{fab}} = 0.000$$

$${}_5F_5\ f_{\text{fab}} = 0.400$$

$${}_5F_{10}\ f_{\text{fab}} = 0.300$$

- Number of survivors to specific age

$$l_0 = 1.000$$

$$l_5 = 0.900$$

$$l_{10} = 0.600$$

$$l_{15} = 0.000$$

Three parts to complete

- We have three parts of the Leslie matrix to fill in
  - Structural zeros
  - Subdiagonal elements
  - First row

Structural zeros

- The first step in writing down the Leslie matrix is to fill in the structural zeros
  - Number 0 go on the structural zeros
- Crosses for nonzero elements go on
  - Subdiagonal
  - First row

$$\begin{pmatrix} X & X & X \\ X & 0 & 0 \\ 0 & X & 0 \end{pmatrix}$$

Subdiagonal

- Our second step is to compute survivorship ratios for the subdiagonal

$$A_{j+1,j} = \frac{{}_nL_{x+n}}{{}_nL_x} = \frac{(5/2)(l_{x+n} + l_{x+n+n})}{(5/2)(l_x + l_{x+n})}$$

$l_0 = 1.000$

$l_5 = 0.900$

$l_{10} = 0.600$

$l_{15} = 0.000$

Age group 0–5: $A_{2\ 1} = \frac{(5/2)(0.9 + 0.6)}{(5/2)(1.0 + 0.9)} = 0.7895$

Age group 5–10: $A_{3\ 2} = \frac{(5/2)(0.6 + 0.0)}{(5/2)(0.9 + 0.6)} = 0.4000$

First row

$$A_{1\ j} = \frac{{}_nL_0}{2\ \ell_0}\left({}_nF_x + {}_nF_{x+n}\frac{{}_nL_{x+n}}{{}_nL_x}\right)(f_{\text{fab}})$$

${}_5F_0\ f_{\text{fab}} = 0.000$ ${}_5L_0 = (5/2)\ (l_0+l_5) = (5/2)\ (1+0.9) = 4.75$

${}_5F_5\ f_{\text{fab}} = 0.400$ ${}_5L_5 = (5/2)\ (l_5+l_{10}) = (5/2)\ (0.9+0.6) = 3.75$

${}_5F_{10}\ f_{\text{fab}} = 0.300$ ${}_5L_{10} = (5/2)\ (l_{10}+l_{15}) = (5/2)\ (0.6+0.0) = 1.50$

Age group 0–5: $A_{1\ 1} = \frac{4.75}{2}(0 + 0.400\frac{3.75}{4.75}) = 0.7500$

Age group 5–10: $A_{1\ 2} = \frac{4.75}{2}(0.400 + 0.300\frac{1.50}{3.75}) = 1.2350$

Age group 10–15: $A_{1\ 3} = \frac{4.75}{2}(0.300 + 0.000) = 0.7125$

Leslie matrix for this example

$$\begin{pmatrix} 0.7500 & 1.2350 & 0.7125 \\ 0.7895 & 0 & 0 \\ 0 & 0.4000 & 0 \end{pmatrix}$$

Use matrix for projection

- Suppose at time *t*=0, we have
  - Zero fillies (0–5)
  - Four mares (5–10)
  - Two seniors (10–15)
- How many fillies, mares, and seniors should we expect after 5 years?

Rule for matrix multiplication

- If we write our population counts at time *t* as a vector *K*(*t*), then we apply the standard rule for matrix multiplication
  - Expected population at time *t* (vector *K*(*t*)) equals the product of matrix *A* times vector *K*(0)
- The general rule for matrix products says that

$K(t) = A\,K(0)$, which means...

$$K_i(t) = \sum A_{i\ j} K_j(0)$$

  - Summing over columns *j* of the matrix and elements *j* of the vector

How many fillies (0–5)?

- First row informs number of fillies to expect at the end of the 5-year step per horse at the start

$$\begin{pmatrix} 0.7500 & 1.2350 & 0.7125 \\ 0.7895 & 0 & 0 \\ 0 & 0.4000 & 0 \end{pmatrix}$$

$$\begin{aligned}&(0.7500 \text{ fillies per starting filly}) * (0 \text{ starting fillies}) \\ &\quad + (1.2350 \text{ fillies per starting mare}) * (4 \text{ starting mares}) \\ &\qquad + (0.7125 \text{ fillies per starting senior}) * (2 \text{ starting seniors})\end{aligned}$$

0 + 4.94 + 1.425 = 6.365 fillies

- We should expect six or seven fillies
  - Around 1/3 chance of having 7 horses
  - Around 2/3 chance of having 6 horses

How many mares (5–10)?

- Use the second row of our matrix
- We go across the columns as we go down the starting vector *K*(0)
  - Zero starting fillies, four starting mares, two starting seniors
- We multiply and add up

$$\begin{pmatrix} 0.7500 & 1.2350 & 0.7125 \\ 0.7895 & 0 & 0 \\ 0 & 0.4000 & 0 \end{pmatrix}$$

$$(0.7895 \text{ mares per starting filly}) * (0 \text{ starting fillies}) + 0 + 0$$

0 mares

How many seniors (10–15)?

- Use the third row of our matrix
- We go across the columns as we go down the starting vector *K*(0)
  - Zero starting fillies, four starting mares, two starting seniors
- We multiply and add up

$$\begin{pmatrix} 0.7500 & 1.2350 & 0.7125 \\ 0.7895 & 0 & 0 \\ 0 & 0.4000 & 0 \end{pmatrix}$$

$$0 + (0.400 \text{ seniors per starting mare}) * (4 \text{ starting mares}) + 0$$

1.6 seniors

Summary

- Equation for population projection

$$K(t) = A\, K(0)$$

- Matrices

$$\begin{array}{ccc} K(t) & \text{Leslie matrix = matrix } A & K(0) \\ \begin{pmatrix} 6.365 \\ 0.000 \\ 1.600 \end{pmatrix} = & \begin{pmatrix} 0.7500 & 1.2350 & 0.7125 \\ 0.7895 & 0 & 0 \\ 0 & 0.4000 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix} \end{array}$$

Matrix notation

- Matrix notation makes it easy to see what happens next

$$K(10) = AK(5) = AAK(0)$$

- $AA = A^2$ is not ordinary multiplication but matrix multiplication

- In general

$$K(nk) = A^k\, K(0)$$

Comparing to previous equation

- Equation from crude model of exponential growth

$$K(T) = A^T K(0)$$

  – It has ordinary numbers rather than vector
  – It does not account for age

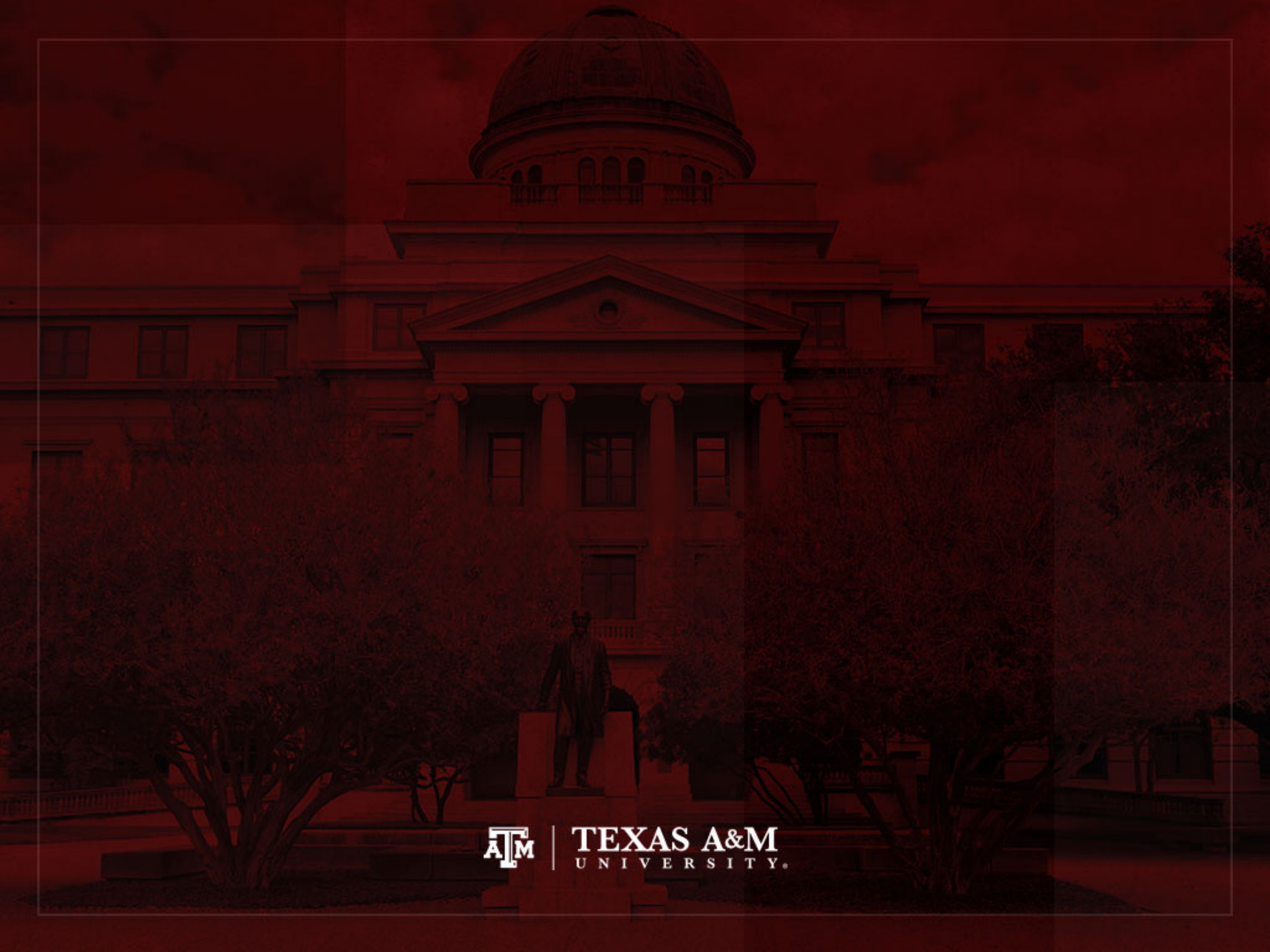
- New equation for population projection

$$K(nk) = A^k K(0)$$

  – It utilizes matrices and vectors, which consider age
  – Same form as before, but new and richer interpretation
  – This is the generalization to age-structured populations of the Crude Rate Model, but it is still a closed population (no migration)

References

Wachter KW. 2014. Essential Demographic Methods. Cambridge: Harvard University Press. Chapter 5 (pp. 98–124).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